Stuff you must know. Cold.

Curve Sketching and Analysis

$y = f(x)$ must be continuous at each:

- Critical Point: $\frac{dy}{dx} = 0$ or undefined. Don't forget endpoints.

- Local Minimum: $\frac{dy}{dx}$ goes $(-,0,+)$ or $(-,\text{undefined},+)$ or $\frac{d^2y}{dx^2} > 0$

- Local Maximum: $\frac{dy}{dx}$ goes $(+,0,-)$ or $(+,\text{undefined},-)$ or $\frac{d^2y}{dx^2} < 0$

- Point of Inflection: Concavity Changes:
    - $\frac{d^2y}{dx^2}$ goes from $(+,0,-)$, $(-,0,+)$, $(+,\text{undefined},-)$, or $(-,\text{undefined},+)$

Derivatives

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

$$D_x\left[k \cdot f(x)\right] = k \cdot D_x f(x)$$

$$D_x\left[f(x)+g(x)\right] = D_x f(x) + D_x g(x)$$

$$D_x\left[f(x)-g(x)\right] = D_x f(x) - D_x g(x)$$

$$D_x\left[f(x)g(x)\right] = f(x)D_x g(x) + D_x f(x)g(x)$$

$$D_x\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)D_x f(x) - f(x)D_x g(x)}{g^2(x)}$$ lodhi minus hidlo over lo squared

$$D_x f(g(x)) = f'(g(x))g'(x)$$ or $$\frac{d}{dx}f(u) = f'(u)\frac{du}{dx}$$

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\tan x) = \sec^2 x$$

$$D_x(\cot x) = -\csc^2 x \qquad D_x(\sec x) = \sec x \tan x \qquad D_x(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx} \qquad \frac{d}{dx}(e^u) = e^u D_x u \qquad \frac{d}{dx}a^x = a^x \ln a D_x x$$

$$\frac{d}{dx}\log_a x = \frac{1}{x\ln a}D_x x$$

$$D_x \sin^{-1} x = \frac{1}{\sqrt{1-x^2}},\ -1 < x < 1 \qquad D_x \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}},\ -1 < x < 1$$

$$D_x \tan^{-1} x = \frac{1}{1+x^2} \qquad D_x \sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}},\ |x| > 1$$

$$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2} \qquad \frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}$$

Integrals **PLUS A CONSTANT!**

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad n \neq -1 \qquad \text{AND} \qquad \int x^n dx = \ln|x| + C \quad n = -1$$

$$\int 1\, dx = x + C \qquad \int kf(x)\, dx = k\int f(x)dx \quad \text{OR} \quad \int kdu = ku + C$$

$$\int [f(x) + g(x)]\, dx = \int f(x)dx + \int g(x)dx$$

$$\int [f(x) - g(x)]\, dx = \int f(x)dx - \int g(x)dx$$

$$\int [f(x)]^r f'(x)dx = \frac{[g(x)]^{r+1}}{r+1} + C$$

$$\int e^u du = e^u + C \qquad \int a^u du = \frac{a^u}{\ln a} + C$$

$$\int \frac{1}{u} du = \ln|u| + C \qquad \int \ln u\, du = u\ln u - u + C$$

$$\int \sin u\, du = -\cos u + C \qquad \int \cos u\, du = \sin u + C$$

$$\int \sec^2 u\, du = \tan u + C \qquad \int \csc^2 u\, du = -\cot u + C$$

$$\int \sec u \tan u \, du = \sec u + C$$

$$\int \csc u \cot u \, du = -\csc u + C$$

$$\int \tan u \, du = -\ln|\cos u| + C$$

$$\int \cot u \, du = \ln|\sin u| + C$$

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{|u|}{a}\right) + C = \frac{1}{a}\cos^{-1}\left(\frac{a}{|u|}\right) + C$$

Transcendentals

$$e^{\ln x} = x \qquad \ln\left(e^y\right) = y \qquad a^x = e^{x \ln a}$$

$$\ln\left(a^x\right) = \ln\left(e^{x\ln a}\right) = x \ln a \qquad \log_a x = \frac{\ln x}{\ln a}$$

$$D_x e^u = e^u D_x u \quad \rightarrow \quad D_x a^x = a^x \ln a \cdot D_x x$$

$$\int e^u du = e^u + C \quad \rightarrow \quad \int a^u du = \frac{a^u}{\ln a} + C$$

$$y = \log_a x \quad \rightarrow \quad x = a^y$$

Fundamental Theorem of Calculus

The distance s travelled from time $t = 0$ to time $t = x$ is $s(x) = \int_0^x f(t)\,dt$. The derivative of this position (velocity) is $s'(x) = v = f(x)$ …or…

$\frac{d}{dx}s(x) = \frac{d}{dx}\int_0^x f(t)\,dt = f(x)$, so the integral is an accumulation function and leads us to:

$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$ or the FTC II

If $F' = f$, then:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$ or the FTC I.

Riemann Sums

$$R_n = \Delta x\left[f(a+\Delta x) + f(a+2\Delta x) + \ldots + f(a+n\Delta x)\right]$$

$$L_n = \Delta x\left[f(a) + f(a+\Delta x) + f(a+2\Delta x) + \ldots + f(a+(n-1)\Delta x)\right]$$

$$M_n = \Delta x\left[f\left(a+\frac{1}{2}\Delta x\right) + f\left(a+\frac{3}{2}\Delta x\right) + \ldots + f\left(a+\left(n-\frac{1}{2}\right)\Delta x\right)\right]$$

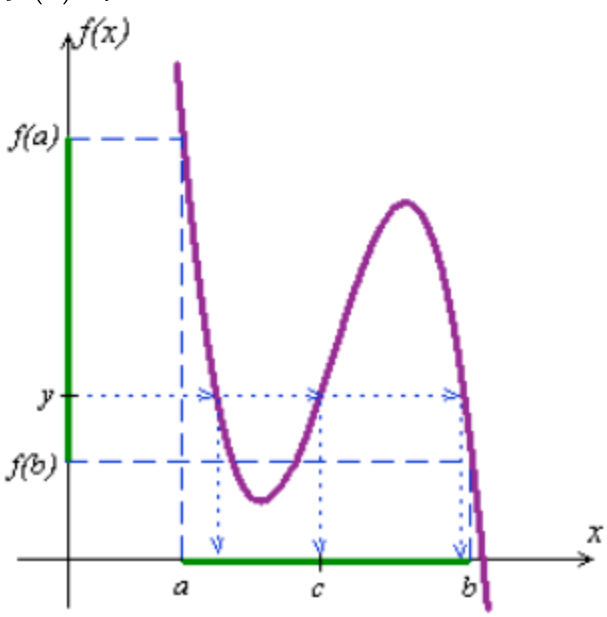
$$T_N \text{ of } \int_a^b f(x)\,dx = \frac{1}{2}\Delta x\left(y_0 + 2y_1 + \ldots + 2y_{n-1} + y_n\right)$$

…or…

$$T_N \text{ of } \int_a^b f(x)\,dx = \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + \ldots 2f(x_{n-1}) + f(x_n)\right]$$

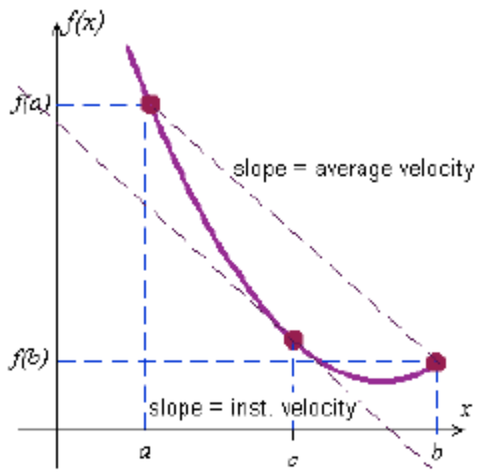
Intermediate Value Theorem

If the function $f(x)$ is continuous on $[a,b]$, and y is a number between $f(a)$ and $f(b)$, then there exists at least one number $x = c$ in the open interval (a,b) such that $f(c) = y$



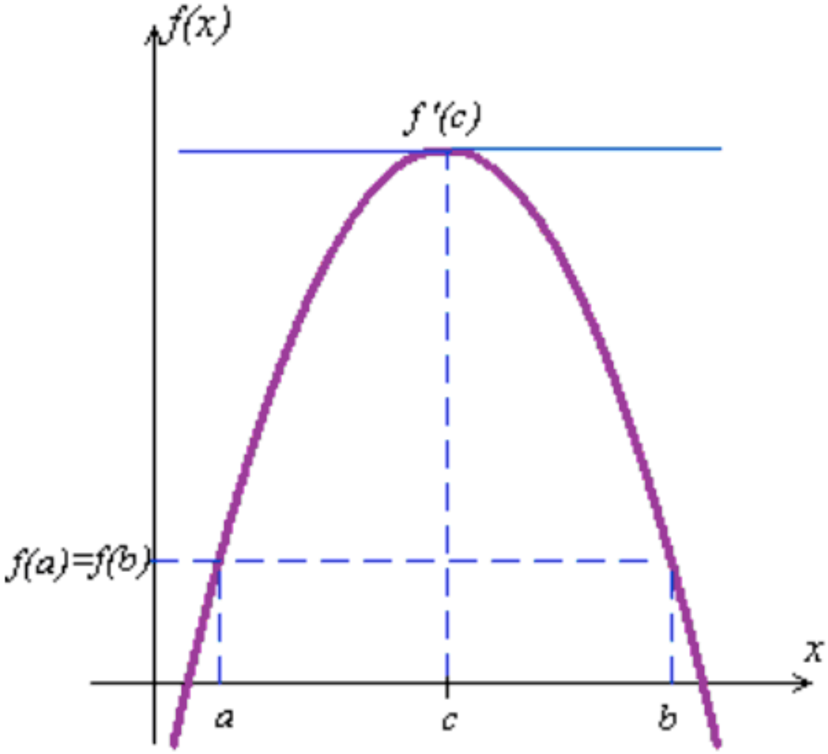
Mean Value Theorem

If the function $f(x)$ is continuous on $[a,b]$, AND the first derivative exists on the interval (a,b), then there is at least one number $x = c$ in (a,b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$



Rolle's Theorem

If the function $f(x)$ is continuous on $[a,b]$, AND the first derivative exists on the interval (a,b), AND $f(a) = f(b)$, then there is at least one number $x = c$ in (a,b) such that $f'(c) = 0$



Theorem of the Mean Value i.e. Average Value

If the function $f\left(x\right)$ is continuous on $\left[a,b\right]$, AND the first derivative exists on the interval $\left(a,b\right)$, then there is at least one number $x=c$ in $\left(a,b\right)$ such that $f\left(c\right)=\dfrac{\int_a^b f\left(x\right)dx}{\left(b-a\right)}$

This value $f\left(c\right)$ is the average value (height) of the function on $\left[a,b\right]$.

Solids of Revolution (Disks, Washers)

Disks: $V=\pi\int_{x=a}^{x=b} r^2 dx$ where r is the function height on $\left[a,b\right]$.

Washers: $V=\pi\int_{x=a}^{x=b}\left(r_2^{\,2}-r_1^{\,2}\right)dx$ where r_2 is the height of the upper function above the axis of revolution and r_1 is the height of the lower function above the axis of revolution.

Distance, Velocity and Acceleration

Velocity $=\dfrac{d}{dt}$(position)

Acceleration $= \frac{d}{dt}$(velocity)

Average Velocity $= \frac{\text{final position - initial position}}{\text{total time}}$ or $\frac{\Delta x}{\Delta t}$

Displacement (directed distance from a starting point) $\int_a^b v(t)\,dt = s(b) - s(a)$

Distance (total distance covered) $\int_a^b |v(t)|\,dt$

L'Hôpital's Rule

If $\lim_{x \to u} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\infty}{\infty}$, then $\lim_{x \to u} \frac{f(x)}{g(x)} = \lim_{x \to u} \frac{f'(x)}{g'(x)}$

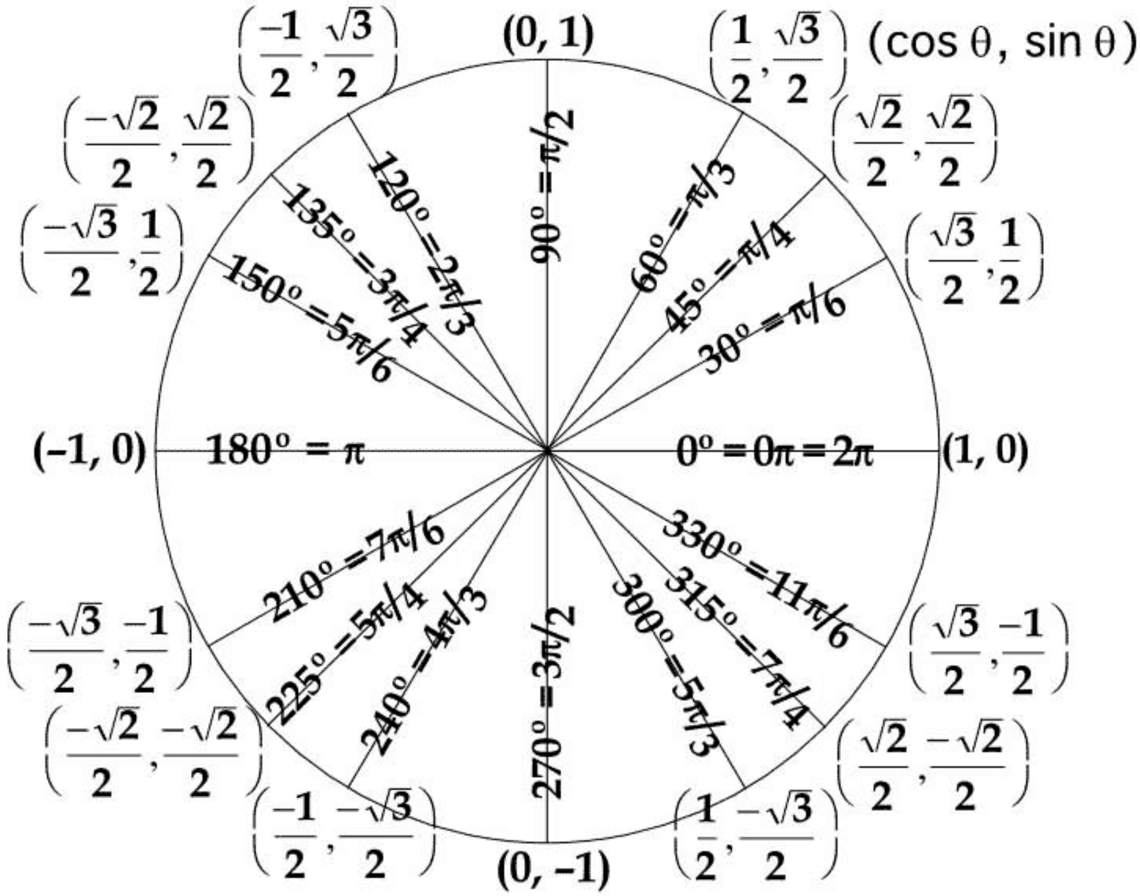
Integration by Parts

LIPET (Log, Inverse, Polynomial, Exponential, Trigonometric)

$\int u\,dv = uv - \int v\,du$

$\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du$

Values of Trig Functions for Common Angles



Trig Identities

Odd-Even Identities

$$\sin(-x) = -\sin x$$

$$\cos(-x) = \cos x$$

$$\tan(-x) = -\tan x$$

Cofunction Identities

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x$$

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x$$

$$\tan\left(\frac{\pi}{2} - x\right) = \cot x$$

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

Addition Identites

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

$$1 + \cot^2 x = \csc^2 x$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

Double-Angle Identities

$$\sin 2x = 2\sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\cos 2x = 2\cos^2 x - 1$$

$$\cos 2x = 1 - 2\sin^2 x$$

Half-Angle Identities

$$\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos x}{2}}$$

$$\cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1+\cos x}{2}}$$

Sum Identities

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

Product Identities

$$\sin x \sin y = -\frac{1}{2}\left[\cos(x+y) - \cos(x-y)\right]$$

$$\cos x \cos y = \frac{1}{2}\left[\cos(x+y) + \cos(x-y)\right]$$

$$\sin x \cos y = \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right]$$

Other Useful Identities

$$\sin\left(\cos^{-1} x\right) = \sqrt{1-x^2}$$

$$\cos\left(\sin^{-1} x\right) = \sqrt{1-x^2}$$

$$\sec\left(\tan^{-1} x\right) = \sqrt{1+x^2}$$

$$\tan\left(\sec^{-1} x\right) = \begin{cases} \sqrt{x^2-1}, & x \geq 1 \\ -\sqrt{x^2-1}, & x \leq -1 \end{cases}$$

Thoughts

Do not simplify arithmetic (even in Riemann sums)
Do not cross out your work (unless you have done it better)
Label your answers
If you think your answer to part a is wrong, use it anyway to continue the problem
Notate your calculator work on the page
Round to three decimal places
Name the function in question: “the slope of f is” not “the function’s slope is”
When writing integrals, make sure that at least the limits and the constant are correct
Know the difference between local (relative) and global (absolute) extrema
Know the difference between the extreme value and the location of the extreme value
Know the difference between a point in time and an interval of time
Be sure that you have answered (not just solved for *x*) the problem with appropriate units
If you can eliminate some wrong answers in the multiple choice section, it is advantageous to guess. If you can not eliminate answers, do not guess. Wrong answers can often be eliminated by estimation or by thinking graphically.
If you are asked to justify your answer, think about what needs justification.

Calculus BC: Section I

Section I consists of 45 multiple-choice questions. Part A contains 28 questions and does not allow the use of a calculator. Part B contains 17 questions and requires a graphing calculator for some questions. Twenty-four sample multiple-choice questions for Calculus BC are included in the following sections. Answers to the sample questions are given on page 39.

Part A Sample Multiple-Choice Questions

A calculator may not be used on this part of the exam.

Part A consists of 28 questions. Following are the directions for Section I, Part A, and a representative set of 14 questions.

Directions: Solve each of the following problems, using the available space for scratch work. After examining the form of the choices, decide which is the best of the choices given and fill in the corresponding circle on the answer sheet. No credit will be given for anything written in the exam book. Do not spend too much time on any one problem.

In this exam:

(1) Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

(2) The inverse of a trigonometric function f may be indicated using the inverse function notation f^{-1} or with the prefix "arc" (e.g., $\sin^{-1} x = \arcsin x$).

1. A curve is described by the parametric equations $x = t^2 + 2t$ and $y = t^3 + t^2$. An equation of the line tangent to the curve at the point determined by $t = 1$ is

(A) $2x - 3y = 0$
(B) $4x - 5y = 2$
(C) $4x - y = 10$
(D) $5x - 4y = 7$
(E) $5x - y = 13$

2. If $3x^2 + 2xy + y^2 = 1$, then $\frac{dy}{dx} =$

(A) $-\frac{3x+y}{y^2}$

(B) $-\frac{3x+y}{x+y}$

(C) $\frac{1-3x-y}{x+y}$

(D) $-\frac{3x}{1+y}$

(E) $-\frac{3x}{x+y}$

x	$g'(x)$
−1.0	2
−0.5	4
0.0	3
0.5	1
1.0	0
1.5	−3
2.0	−6

3. The table above gives selected values for the derivative of a function g on the interval $-1 \le x \le 2$. If $g(-1) = -2$ and Euler's method with a step-size of 1.5 is used to approximate $g(2)$, what is the resulting approximation?

(A) −6.5
(B) −1.5
(C) 1.5
(D) 2.5
(E) 3

4. What are all values of x for which the series $\sum_{n=1}^{\infty} \frac{n3^n}{x^n}$ converges?

(A) All x except $x = 0$
(B) $|x| = 3$
(C) $-3 \le x \le 3$
(D) $|x| > 3$
(E) The series diverges for all x.

5. If $\frac{d}{dx}f(x) = g(x)$ and if $h(x) = x^2$, then $\frac{d}{dx}f(h(x)) =$

(A) $g\left(x^2\right)$

(B) $2xg(x)$

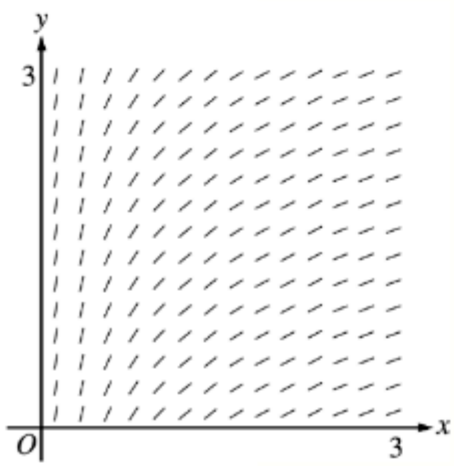
(C) $g'(x)$

(D) $2xg\left(x^2\right)$

(E) $x^2g\left(x^2\right)$

6. If F' is a continuous function for all real x, then $\lim_{h\to 0}\frac{1}{h}\int_a^{a+h} F'(x)\,dx$ is

(A) 0
(B) $F(0)$
(C) $F(a)$
(D) $F'(0)$
(E) $F'(a)$



7. The slope field for a certain differential equation is shown above. Which of the following could be a specific solution to that differential equation?

(A) $y = x^2$
(B) $y = e^x$
(C) $y = e^{-x}$
(D) $y = \cos x$
(E) $y = \ln x$

8. $\int_0^3 \frac{dx}{(1-x)^2}$ is

(A) $-\frac{3}{2}$

(B) $-\frac{1}{2}$

(C) $\frac{1}{2}$

(D) $\frac{3}{2}$

(E) divergent

9. Which of the following series converge to 2?

I. $\sum_{n=1}^{\infty} \frac{2n}{n+3}$

II. $\sum_{n=1}^{\infty} \frac{-8}{(-3)^n}$

III. $\sum_{n=0}^{\infty} \frac{1}{2^n}$

(A) I only
(B) II only
(C) III only
(D) I and III only
(E) II and III only

10. If the function f given by $f(x) = x^3$ has an average value of 9 on the closed interval $[0, k]$, then $k =$

(A) 3

(B) $3^{1/2}$

(C) $18^{1/3}$

(D) $36^{1/4}$

(E) $36^{1/3}$

11. Which of the following integrals gives the length of the graph $y = \sin(\sqrt{x})$ between $x = a$ and $x = b$, where $0 < a < b$?

(A) $\int_a^b \sqrt{x + \cos^2(\sqrt{x})}\, dx$

(B) $\int_a^b \sqrt{1 + \cos^2(\sqrt{x})}\, dx$

(C) $\int_a^b \sqrt{\sin^2(\sqrt{x}) + \frac{1}{4x}\cos^2(\sqrt{x})}\, dx$

(D) $\int_a^b \sqrt{1 + \frac{1}{4x}\cos^2(\sqrt{x})}\, dx$

(E) $\int_a^b \sqrt{\frac{1 + \cos^2(\sqrt{x})}{4x}}\, dx$

12. Which of the following integrals represents the area enclosed by the smaller loop of the graph of $r = 1 + 2\sin\theta$?

(A) $\frac{1}{2}\int_{7\pi/6}^{11\pi/6} (1 + 2\sin\theta)^2\, d\theta$

(B) $\frac{1}{2}\int_{7\pi/6}^{11\pi/6} (1 + 2\sin\theta)\, d\theta$

(C) $\frac{1}{2}\int_{-\pi/6}^{7\pi/6} (1 + 2\sin\theta)^2\, d\theta$

(D) $\int_{-\pi/6}^{7\pi/6} (1 + 2\sin\theta)^2\, d\theta$

(E) $\int_{7\pi/6}^{-\pi/6} (1 + 2\sin\theta)\, d\theta$

13. The third-degree Taylor polynomial about $x = 0$ of $\ln(1 - x)$ is

(A) $-x - \frac{x^2}{2} - \frac{x^3}{3}$

(B) $1 - x + \frac{x^2}{2}$

(C) $x - \frac{x^2}{2} + \frac{x^3}{3}$

(D) $-1 + x - \frac{x^2}{2}$

(E) $-x + \frac{x^2}{2} - \frac{x^3}{3}$

14. If $\frac{dy}{dx} = y\sec^2 x$ and $y = 5$ when $x = 0$, then $y =$

(A) $e^{\tan x} + 4$

(B) $e^{\tan x} + 5$

(C) $5e^{\tan x}$

(D) $\tan x + 5$

(E) $\tan x + 5e^x$

Part B Sample Multiple-Choice Questions

A graphing calculator is required for some questions on this part of the exam.

Part B consists of 17 questions. Following are the directions for Section I, Part B, and a representative set of 10 questions.

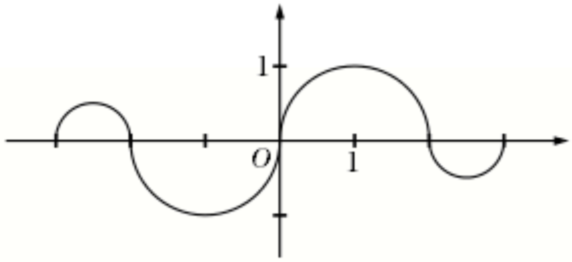
Directions: Solve each of the following problems, using the available space for scratch work. After examining the form of the choices, decide which is the best of the choices given and fill in the corresponding circle on the answer sheet. No credit will be given for anything written in the exam book. Do not spend too much time on any one problem.

In this exam:

(1) The exact numerical value of the correct answer does not always appear among the choices given. When this happens, select from among the choices the number that best approximates the exact numerical value.

(2) Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

(3) The inverse of a trigonometric function f may be indicated using the inverse function notation f^{-1} or with the prefix "arc" (e.g., $\sin^{-1} x = \arcsin x$).



Graph of f

15. The graph of the function f above consists of four semicircles. If $g(x) = \int_0^x f(t)\,dt$, where is $g(x)$ nonnegative?

(A) $[-3, 3]$
(B) $[-3, -2] \cup [0, 2]$ only
(C) $[0, 3]$ only
(D) $[0, 2]$ only
(E) $[-3, -2] \cup [0, 3]$ only

16. If f is differentiable at $x = a$, which of the following could be false?

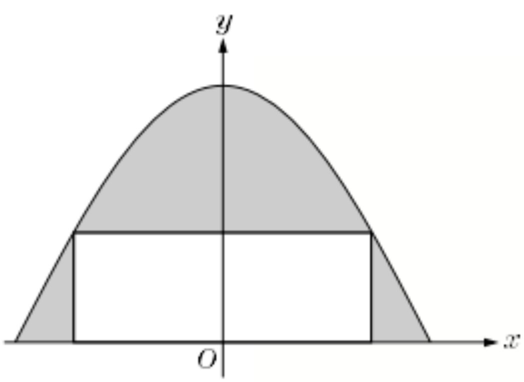
(A) f is continuous at $x = a$.

(B) $\lim_{x \to a} f(x)$ exists.

(C) $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists.

(D) $f'(a)$ is defined.

(E) $f''(a)$ is defined.



17. A rectangle with one side on the x-axis has its upper vertices on the graph of $y = \cos x$, as shown in the figure above. What is the minimum area of the shaded region?

(A) 0.799
(B) 0.878
(C) 1.140
(D) 1.439
(E) 2.000

18. A solid has a rectangular base that lies in the first quadrant and is bounded by the x- and y-axes and the lines $x = 2$ and $y = 1$. The height of the solid above the point (x, y) is $1 + 3x$. Which of the following is a Riemann sum approximation for the volume of the solid?

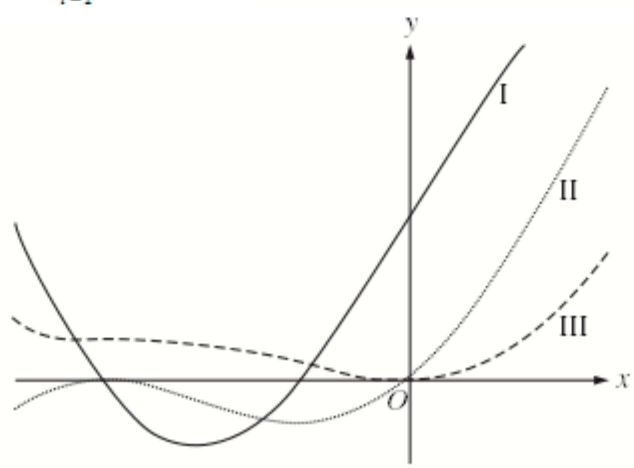
(A) $\sum_{i=1}^{n} \frac{1}{n}\left(1 + \frac{3i}{n}\right)$

(B) $2\sum_{i=1}^{n} \frac{1}{n}\left(1 + \frac{3i}{n}\right)$

(C) $2\sum_{i=1}^{n} \frac{i}{n}\left(1 + \frac{3i}{n}\right)$

(D) $\sum_{i=1}^{n} \frac{2}{n}\left(1 + \frac{6i}{n}\right)$

(E) $\sum_{i=1}^{n} \frac{2i}{n}\left(1 + \frac{6i}{n}\right)$



19. Three graphs labeled I, II, and III are shown above. One is the graph of f, one is the graph of f', and one is the graph of f''. Which of the following correctly identifies each of the three graphs?

	f	f'	f''
(A)	I	II	III
(B)	I	III	II
(C)	II	I	III
(D)	II	III	I
(E)	III	II	I

20. A particle moves along the x-axis so that at any time $t \geq 0$ its velocity is given by $v(t) = \ln(t + 1) - 2t + 1$. The total distance traveled by the particle from $t = 0$ to $t = 2$ is

(A) 0.667
(B) 0.704
(C) 1.540
(D) 2.667
(E) 2.901

21. If the function f is defined by $f(x) = \sqrt{x^3 + 2}$ and g is an antiderivative of f such that $g(3) = 5$, then $g(1) =$

(A) -3.268
(B) -1.585
(C) 1.732
(D) 6.585
(E) 11.585

22. Let g be the function given by $g(x) = \int_1^x 100\left(t^2 - 3t + 2\right)e^{-t^2}\,dt$.

Which of the following statements about g must be true?

I. g is increasing on $(1, 2)$.

II. g is increasing on $(2, 3)$.

III. $g(3) > 0$

(A) I only
(B) II only
(C) III only
(D) II and III only
(E) I, II, and III

23. For a series S, let

$$S = 1 - \frac{1}{9} + \frac{1}{2} - \frac{1}{25} + \frac{1}{4} - \frac{1}{49} + \frac{1}{8} - \frac{1}{81} + \frac{1}{16} - \frac{1}{121} + \cdots + a_n + \cdots,$$

$$\text{where } a_n = \begin{cases} \dfrac{1}{2^{(n-1)/2}} & \text{if } n \text{ is odd} \\ \dfrac{-1}{(n+1)^2} & \text{if } n \text{ is even.} \end{cases}$$

Which of the following statements are true?

I. S converges because the terms of S alternate and $\lim_{n\to\infty} a_n = 0$.

II. S diverges because it is not true that $|a_{n+1}| < |a_n|$ for all n.

III. S converges although it is not true that $|a_{n+1}| < |a_n|$ for all n.

(A) None
(B) I only
(C) II only
(D) III only
(E) I and III only

24. Let g be the function given by $g(t) = 100 + 20\sin\left(\frac{\pi t}{2}\right) + 10\cos\left(\frac{\pi t}{6}\right)$.

For $0 \le t \le 8$, g is decreasing most rapidly when $t =$

(A) 0.949
(B) 2.017
(C) 3.106
(D) 5.965
(E) 8.000

Answers to Calculus BC Multiple-Choice Questions

Part A		*Part B*	
†1.	D	15.	A
2.	B	16.	E
†3.	D	17.*	B
†4.	D	18.	D
5.	D	19.	E
6.	E	20.*	C
7.	E	21.*	B
†8.	E	22.*	B
†9.	E	†23.	D
10.	E	24.*	B
†11.	D		
†12.	A		
†13.	A		
14.	C		

90 Minutes—No Calculator

Note: In this examination, $\ln x$ denotes the natural logarithm of x (that is, logarithm to the base e).

1. The asymptotes of the graph of the parametric equations $x=\frac{1}{t}$, $y=\frac{t}{t+1}$ are

(A) $x=0$, $y=0$ (B) $x=0$ only (C) $x=-1$, $y=0$
(D) $x=-1$ only (E) $x=0$, $y=1$

---

2. What are the coordinates of the inflection point on the graph of $y=(x+1)\arctan x$?

(A) $(-1,0)$ (B) $(0,0)$ (C) $(0,1)$ (D) $\left(1,\frac{\pi}{4}\right)$ (E) $\left(1,\frac{\pi}{2}\right)$

---

3. The Mean Value Theorem guarantees the existence of a special point on the graph of $y=\sqrt{x}$ between $(0,0)$ and $(4,2)$. What are the coordinates of this point?

(A) $(2,1)$
(B) $(1,1)$
(C) $(2,\sqrt{2})$
(D) $\left(\frac{1}{2},\frac{1}{\sqrt{2}}\right)$
(E) None of the above

---

4. $\int_0^8 \frac{dx}{\sqrt{1+x}}=$

(A) 1 (B) $\frac{3}{2}$ (C) 2 (D) 4 (E) 6

---

5. If $3x^2+2xy+y^2=2$, then the value of $\frac{dy}{dx}$ at $x=1$ is

(A) -2 (B) 0 (C) 2 (D) 4 (E) not defined

6. What is $\lim_{h\to 0}\dfrac{8\left(\frac{1}{2}+h\right)^8-8\left(\frac{1}{2}\right)^8}{h}$?

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) The limit does not exist.

(E) It cannot be determined from the information given.

---

7. For what value of *k* will $x+\frac{k}{x}$ have a relative maximum at $x=-2$?

(A) –4 (B) –2 (C) 2 (D) 4 (E) None of these

---

8. If $h(x)=f^2(x)-g^2(x)$, $f'(x)=-g(x)$, and $g'(x)=f(x)$, then $h'(x)=$

(A) 0 (B) 1 (C) $-4f(x)g(x)$

(D) $(-g(x))^2-(f(x))^2$ (E) $-2(-g(x)+f(x))$

---

9. The area of the closed region bounded by the polar graph of $r=\sqrt{3+\cos\theta}$ is given by the integral

(A) $\int_0^{2\pi}\sqrt{3+\cos\theta}\,d\theta$ (B) $\int_0^{\pi}\sqrt{3+\cos\theta}\,d\theta$ (C) $2\int_0^{\pi/2}(3+\cos\theta)\,d\theta$

(D) $\int_0^{\pi}(3+\cos\theta)\,d\theta$ (E) $2\int_0^{\pi/2}\sqrt{3+\cos\theta}\,d\theta$

---

10. $\int_0^1\frac{x^2}{x^2+1}dx=$

(A) $\frac{4-\pi}{4}$ (B) $\ln 2$ (C) 0 (D) $\frac{1}{2}\ln 2$ (E) $\frac{4+\pi}{4}$

11. The point <u>on the curve</u> $x^2 + 2y = 0$ that is nearest the point $\left(0, -\frac{1}{2}\right)$ occurs where y is

(A) $\frac{1}{2}$

(B) 0

(C) $-\frac{1}{2}$

(D) -1

(E) none of the above

---

12. If $F(x) = \int_0^x e^{-t^2}\,dt$, then $F'(x) =$

(A) $2xe^{-x^2}$ (B) $-2xe^{-x^2}$ (C) $\frac{e^{-x^2+1}}{-x^2+1} - e$

(D) $e^{-x^2} - 1$ (E) e^{-x^2}

---

13. The region bounded by the x-axis and the part of the graph of $y = \cos x$ between $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$ is separated into two regions by the line $x = k$. If the area of the region for $-\frac{\pi}{2} \le x \le k$ is three times the area of the region for $k \le x \le \frac{\pi}{2}$, then $k =$

(A) $\arcsin\left(\frac{1}{4}\right)$ (B) $\arcsin\left(\frac{1}{3}\right)$ (C) $\frac{\pi}{6}$ (D) $\frac{\pi}{4}$ (E) $\frac{\pi}{3}$

---

14. If $y = x^2 + 2$ and $u = 2x - 1$, then $\frac{dy}{du} =$

(A) $\frac{2x^2 - 2x + 4}{(2x-1)^2}$ (B) $6x^2 - 2x + 4$ (C) x^2

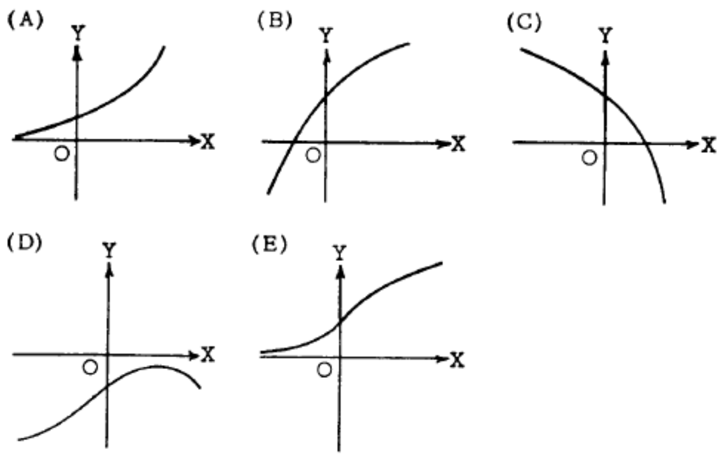
(D) x (E) $\frac{1}{x}$

15. If $f'(x)$ and $g'(x)$ exist and $f'(x) > g'(x)$ for all real x, then the graph of $y = f(x)$ and the graph of $y = g(x)$

(A) intersect exactly once.
(B) intersect no more than once.
(C) do not intersect.
(D) could intersect more than once.
(E) have a common tangent at each point of intersection.

---

16. If y is a function x such that $y' > 0$ for all x and $y'' < 0$ for all x, which of the following could be part of the graph of $y = f(x)$?



---

17. The graph of $y = 5x^4 - x^5$ has a point of inflection at

(A) $(0,0)$ only (B) $(3,162)$ only (C) $(4,256)$ only
(D) $(0,0)$ and $(3,162)$ (E) $(0,0)$ and $(4,256)$

---

18. If $f(x) = 2 + |x-3|$ for all x, then the value of the derivative $f'(x)$ at $x = 3$ is

(A) -1 (B) 0 (C) 1 (D) 2 (E) nonexistent

19. A point moves on the x-axis in such a way that its velocity at time t $(t > 0)$ is given by $v = \frac{\ln t}{t}$.
At what value of t does v attain its maximum?

(A) 1 (B) $e^{\frac{1}{2}}$ (C) e (D) $e^{\frac{3}{2}}$

(E) There is no maximum value for v.

---

20. An equation for a tangent to the graph of $y = \arcsin \frac{x}{2}$ at the origin is

(A) $x - 2y = 0$ (B) $x - y = 0$ (C) $x = 0$

(D) $y = 0$ (E) $\pi x - 2y = 0$

---

21. At $x = 0$, which of the following is true of the function f defined by $f(x) = x^2 + e^{-2x}$?

(A) f is increasing.
(B) f is decreasing.
(C) f is discontinuous.
(D) f has a relative minimum.
(E) f has a relative maximum.

---

22. If $f(x) = \int_0^x \frac{1}{\sqrt{t^3 + 2}} dt$, which of the following is FALSE?

(A) $f(0) = 0$

(B) f is continuous at x for all $x \geq 0$.

(C) $f(1) > 0$

(D) $f'(1) = \frac{1}{\sqrt{3}}$

(E) $f(-1) > 0$

23. If the graph of $y = f(x)$ contains the point $(0, 2)$, $\frac{dy}{dx} = \frac{-x}{ye^{x^2}}$ and $f(x) > 0$ for all x, then $f(x) =$

(A) $3+e^{-x^2}$ (B) $\sqrt{3}+e^{-x}$ (C) $1+e^{-x}$

(D) $\sqrt{3+e^{-x^2}}$ (E) $\sqrt{3+e^{x^2}}$

---

24. If $\sin x = e^y$, $0 < x < \pi$, what is $\frac{dy}{dx}$ in terms of x ?

(A) $-\tan x$ (B) $-\cot x$ (C) $\cot x$ (D) $\tan x$ (E) $\csc x$

---

25. A region in the plane is bounded by the graph of $y = \frac{1}{x}$, the x-axis, the line $x = m$, and the line $x = 2m$, $m > 0$. The area of this region

(A) is independent of m .

(B) increases as m increases.

(C) decreases as m increases.

(D) decreases as m increases when $m < \frac{1}{2}$; increases as m increases when $m > \frac{1}{2}$.

(E) increases as m increases when $m < \frac{1}{2}$; decreases as m increases when $m > \frac{1}{2}$.

---

26. $\int_0^1 \sqrt{x^2 - 2x + 1}\, dx$ is

(A) -1

(B) $-\frac{1}{2}$

(C) $\frac{1}{2}$

(D) 1

(E) none of the above

27. If $\frac{dy}{dx} = \tan x$, then $y =$

(A) $\frac{1}{2}\tan^2 x + C$ (B) $\sec^2 x + C$ (C) $\ln|\sec x| + C$

(D) $\ln|\cos x| + C$ (E) $\sec x \tan x + C$

---

28. What is $\lim_{x \to 0} \frac{e^{2x} - 1}{\tan x}$?

(A) -1 (B) 0 (C) 1 (D) 2 (E) The limit does not exist.

---

29. $\int_0^1 \left(4 - x^2\right)^{-\frac{3}{2}} dx =$

(A) $\frac{2 - \sqrt{3}}{3}$ (B) $\frac{2\sqrt{3} - 3}{4}$ (C) $\frac{\sqrt{3}}{12}$ (D) $\frac{\sqrt{3}}{3}$ (E) $\frac{\sqrt{3}}{2}$

---

30. $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$ is the Taylor series about zero for which of the following functions?

(A) $\sin x$ (B) $\cos x$ (C) e^x (D) e^{-x} (E) $\ln(1+x)$

---

31. If $f'(x) = -f(x)$ and $f(1) = 1$, then $f(x) =$

(A) $\frac{1}{2}e^{-2x+2}$ (B) e^{-x-1} (C) e^{1-x} (D) e^{-x} (E) $-e^x$

---

32. For what values of x does the series $1 + 2^x + 3^x + 4^x + \cdots + n^x + \cdots$ converge?

(A) No values of x (B) $x < -1$ (C) $x \geq -1$ (D) $x > -1$ (E) All values of x

---

33. What is the average (mean) value of $3t^3 - t^2$ over the interval $-1 \leq t \leq 2$?

(A) $\frac{11}{4}$ (B) $\frac{7}{2}$ (C) 8 (D) $\frac{33}{4}$ (E) 16

34. Which of the following is an equation of a curve that intersects at right angles every curve of the family $y = \frac{1}{x} + k$ (where k takes all real values)?

(A) $y = -x$ (B) $y = -x^2$ (C) $y = -\frac{1}{3}x^3$ (D) $y = \frac{1}{3}x^3$ (E) $y = \ln x$

---

35. At $t = 0$ a particle starts at rest and moves along a line in such a way that at time t its acceleration is $24t^2$ feet per second per second. Through how many feet does the particle move during the first 2 seconds?

(A) 32 (B) 48 (C) 64 (D) 96 (E) 192

---

36. The approximate value of $y = \sqrt{4 + \sin x}$ at $x = 0.12$, obtained from the tangent to the graph at $x = 0$, is

(A) 2.00 (B) 2.03 (C) 2.06 (D) 2.12 (E) 2.24

---

37. Of the following choices of δ, which is the largest that could be used successfully with an arbitrary ε in an epsilon-delta proof of $\lim_{x \to 2}(1 - 3x) = -5$?

(A) $\delta = 3\varepsilon$ (B) $\delta = \varepsilon$ (C) $\delta = \frac{\varepsilon}{2}$ (D) $\delta = \frac{\varepsilon}{4}$ (E) $\delta = \frac{\varepsilon}{5}$

---

38. If $f(x) = \left(x^2 + 1\right)^{(2-3x)}$, then $f'(1) =$

(A) $-\frac{1}{2}\ln(8e)$ (B) $-\ln(8e)$ (C) $-\frac{3}{2}\ln(2)$ (D) $-\frac{1}{2}$ (E) $\frac{1}{8}$

---

39. If $y = \tan u$, $u = v - \frac{1}{v}$, and $v = \ln x$, what is the value of $\frac{dy}{dx}$ at $x = e$?

(A) 0 (B) $\frac{1}{e}$ (C) 1 (D) $\frac{2}{e}$ (E) $\sec^2 e$

40. If n is a non-negative integer, then $\int_0^1 x^n\,dx = \int_0^1 (1-x)^n\,dx$ for

(A) no n
(B) n even, only
(C) n odd, only
(D) nonzero n, only
(E) all n

---

41. If $\begin{cases} f(x) = 8 - x^2 & \text{for } -2 \le x \le 2, \\ f(x) = x^2 & \text{elsewhere,} \end{cases}$ then $\int_{-1}^{3} f(x)\,dx$ is a number between

(A) 0 and 8
(B) 8 and 16
(C) 16 and 24
(D) 24 and 32
(E) 32 and 40

---

42. If $\int x^2 \cos x\,dx = f(x) - \int 2x \sin x\,dx$, then $f(x) =$

(A) $2\sin x + 2x\cos x + C$

(B) $x^2 \sin x + C$

(C) $2x\cos x - x^2 \sin x + C$

(D) $4\cos x - 2x\sin x + C$

(E) $\left(2 - x^2\right)\cos x - 4\sin x + C$

---

43. Which of the following integrals gives the length of the graph of $y = \tan x$ between $x = a$ and $x = b$, where $0 < a < b < \dfrac{\pi}{2}$?

(A) $\int_a^b \sqrt{x^2 + \tan^2 x}\,dx$

(B) $\int_a^b \sqrt{x + \tan x}\,dx$

(C) $\int_a^b \sqrt{1 + \sec^2 x}\,dx$

(D) $\int_a^b \sqrt{1 + \tan^2 x}\,dx$

(E) $\int_a^b \sqrt{1 + \sec^4 x}\,dx$

44. If $f''(x)-f'(x)-2f(x)=0$, $f'(0)=-2$, and $f(0)=2$, then $f(1)=$

(A) e^2+e^{-1} (B) 1 C) 0 (D) e^2 (E) $2e^{-1}$

---

45. The complete interval of convergence of the series $\sum_{k=1}^{\infty}\frac{(x+1)^k}{k^2}$ is

(A) $0<x<2$ (B) $0\le x\le 2$ (C) $-2<x\le 0$

(D) $-2\le x<0$ (E) $-2\le x\le 0$

1969 BC

1. C
2. E
3. B
4. D
5. E
6. B
7. D
8. C
9. D
10. A
11. B
12. E
13. C
14. D
15. B
16. B
17. B
18. E
19. C
20. A
21. B
22. E
23. D
24. C
25. A
26. C
27. C
28. D
29. C
30. D
31. C
32. B
33. A
34. D
35. A
36. B
37. D
38. A
39. D
40. E
41. D
42. B
43. E
44. E
45. E

90 Minutes—No Calculator

Note: In this examination, ln x denotes the natural logarithm of x (that is, logarithm to the base e).

1. If $f(x) = e^{1/x}$, then $f'(x) =$

(A) $-\dfrac{e^{1/x}}{x^2}$ (B) $-e^{1/x}$ (C) $\dfrac{e^{1/x}}{x}$ (D) $\dfrac{e^{1/x}}{x^2}$ (E) $\dfrac{1}{x}e^{(1/x)-1}$

---

2. $\int_0^3 (x+1)^{1/2}\,dx =$

(A) $\dfrac{21}{2}$ (B) 7 (C) $\dfrac{16}{3}$ (D) $\dfrac{14}{3}$ (E) $-\dfrac{1}{4}$

---

3. If $f(x) = x + \dfrac{1}{x}$, then the set of values for which f increases is

(A) $(-\infty, -1] \cup [1, \infty)$ (B) $[-1, 1]$ (C) $(-\infty, \infty)$

(D) $(0, \infty)$ (E) $(-\infty, 0) \cup (0, \infty)$

---

4. For what non-negative value of b is the line given by $y = -\dfrac{1}{3}x + b$ normal to the curve $y = x^3$?

(A) 0 (B) 1 (C) $\dfrac{4}{3}$ (D) $\dfrac{10}{3}$ (E) $\dfrac{10\sqrt{3}}{3}$

---

5. $\int_{-1}^{2} \dfrac{|x|}{x}\,dx$ is

(A) -3 (B) 1 (C) 2 (D) 3 (E) nonexistent

---

6. If $f(x) = \dfrac{x-1}{x+1}$ for all $x \neq -1$, then $f'(1) =$

(A) -1 (B) $-\dfrac{1}{2}$ (C) 0 (D) $\dfrac{1}{2}$ (E) 1

7. If $y = \ln\left(x^2 + y^2\right)$, then the value of $\frac{dy}{dx}$ at the point $(1,0)$ is

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) 2 (E) undefined

---

8. If $y = \sin x$ and $y^{(n)}$ means "the nth derivative of y with respect to x," then the smallest positive integer n for which $y^{(n)} = y$ is

(A) 2 (B) 4 (C) 5 (D) 6 (E) 8

---

9. If $y = \cos^2 3x$, then $\frac{dy}{dx} =$

(A) $-6\sin 3x \cos 3x$ (B) $-2\cos 3x$ (C) $2\cos 3x$

(D) $6\cos 3x$ (E) $2\sin 3x \cos 3x$

---

10. The length of the curve $y = \ln \sec x$ from $x = 0$ to $x = b$, where $0 < b < \frac{\pi}{2}$, may be expressed by which of the following integrals?

(A) $\int_0^b \sec x \, dx$

(B) $\int_0^b \sec^2 x \, dx$

(C) $\int_0^b (\sec x \tan x) \, dx$

(D) $\int_0^b \sqrt{1 + (\ln \sec x)^2} \, dx$

(E) $\int_0^b \sqrt{1 + \left(\sec^2 x \tan^2 x\right)} \, dx$

---

11. Let $y = x\sqrt{1 + x^2}$. When $x = 0$ and $dx = 2$, the value of dy is

(A) -2 (B) -1 (C) 0 (D) 1 (E) 2

12. If n is a known positive integer, for what value of k is $\int_1^k x^{n-1}\,dx = \frac{1}{n}$?

(A) 0 (B) $\left(\frac{2}{n}\right)^{1/n}$ (C) $\left(\frac{2n-1}{n}\right)^{1/n}$

(D) $2^{1/n}$ (E) 2^n

---

13. The acceleration α of a body moving in a straight line is given in terms of time t by $\alpha = 8 - 6t$. If the velocity of the body is 25 at $t = 1$ and if $s(t)$ is the distance of the body from the origin at time t, what is $s(4) - s(2)$?

(A) 20 (B) 24 (C) 28 (D) 32 (E) 42

---

14. If $x = t^2 - 1$ and $y = 2e^t$, then $\frac{dy}{dx} =$

(A) $\frac{e^t}{t}$ (B) $\frac{2e^t}{t}$ (C) $\frac{e^{|t|}}{t^2}$ (D) $\frac{4e^t}{2t-1}$ (E) e^t

---

15. The area of the region bounded by the lines $x = 0$, $x = 2$, and $y = 0$ and the curve $y = e^{x/2}$ is

(A) $\frac{e-1}{2}$ (B) $e-1$ (C) $2(e-1)$ (D) $2e-1$ (E) $2e$

---

16. A series expansion of $\frac{\sin t}{t}$ is

(A) $1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!} + \cdots$

(B) $\frac{1}{t} - \frac{t}{2!} + \frac{t^3}{4!} - \frac{t^5}{6!} + \cdots$

(C) $1 + \frac{t^2}{3!} + \frac{t^4}{5!} + \frac{t^6}{7!} + \cdots$

(D) $\frac{1}{t} + \frac{t}{2!} + \frac{t^3}{4!} + \frac{t^5}{6!} + \cdots$

(E) $t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots$

17. The number of bacteria in a culture is growing at a rate of $3,000e^{2t/5}$ per unit of time t. At $t = 0$, the number of bacteria present was 7,500. Find the number present at $t = 5$.

(A) $1,200e^2$ (B) $3,000e^2$ (C) $7,500e^2$ (D) $7,500e^5$ (E) $\frac{15,000}{7}e^7$

---

18. Let g be a continuous function on the closed interval $[0,1]$. Let $g(0) = 1$ and $g(1) = 0$. Which of the following is NOT necessarily true?

(A) There exists a number h in $[0,1]$ such that $g(h) \geq g(x)$ for all x in $[0,1]$.

(B) For all a and b in $[0,1]$, if $a = b$, then $g(a) = g(b)$.

(C) There exists a number h in $[0,1]$ such that $g(h) = \frac{1}{2}$.

(D) There exists a number h in $[0,1]$ such that $g(h) = \frac{3}{2}$.

(E) For all h in the open interval $(0,1)$, $\lim_{x \to h} g(x) = g(h)$.

---

19. Which of the following series converge?

I. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ II. $\sum_{n=1}^{\infty} \frac{1}{n}$ III. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

(A) I only (B) III only (C) I and II only (D) I and III only (E) I, II, and III

---

20. $\int x\sqrt{4-x^2}\,dx =$

(A) $\frac{\left(4-x^2\right)^{3/2}}{3} + C$ (B) $-\left(4-x^2\right)^{3/2} + C$ (C) $\frac{x^2\left(4-x^2\right)^{3/2}}{3} + C$

(D) $-\frac{x^2\left(4-x^2\right)^{3/2}}{3} + C$ (E) $-\frac{\left(4-x^2\right)^{3/2}}{3} + C$

---

21. $\int_0^1 (x+1)e^{x^2+2x}\,dx =$

(A) $\frac{e^3}{2}$ (B) $\frac{e^3-1}{2}$ (C) $\frac{e^4-e}{2}$ (D) e^3-1 (E) e^4-e

22. A particle moves on the curve $y = \ln x$ so that the x-component has velocity $x'(t) = t + 1$ for $t \geq 0$. At time $t = 0$, the particle is at the point $(1, 0)$. At time $t = 1$, the particle is at the point

(A) $(2, \ln 2)$ (B) $(e^2, 2)$ (C) $\left(\frac{5}{2}, \ln\frac{5}{2}\right)$

(D) $(3, \ln 3)$ (E) $\left(\frac{3}{2}, \ln\frac{3}{2}\right)$

---

23. $\lim\limits_{h \to 0} \frac{1}{h} \ln\left(\frac{2+h}{2}\right)$ is

(A) e^2 (B) 1 (C) $\frac{1}{2}$ (D) 0 (E) nonexistent

---

24. Let $f(x) = 3x + 1$ for all real x and let $\varepsilon > 0$. For which of the following choices of δ is $|f(x) - 7| < \varepsilon$ whenever $|x - 2| < \delta$?

(A) $\frac{\varepsilon}{4}$ (B) $\frac{\varepsilon}{2}$ (C) $\frac{\varepsilon}{\varepsilon + 1}$ (D) $\frac{\varepsilon + 1}{\varepsilon}$ (E) 3ε

---

25. $\int_0^{\pi/4} \tan^2 x \, dx =$

(A) $\frac{\pi}{4} - 1$ (B) $1 - \frac{\pi}{4}$ (C) $\frac{1}{3}$ (D) $\sqrt{2} - 1$ (E) $\frac{\pi}{4} + 1$

---

26. Which of the following is true about the graph of $y = \ln\left|x^2 - 1\right|$ in the interval $(-1, 1)$?

(A) It is increasing.
(B) It attains a relative minimum at $(0, 0)$.
(C) It has a range of all real numbers.
(D) It is concave down.
(E) It has an asymptote of $x = 0$.

---

27. If $f(x) = \frac{1}{3}x^3 - 4x^2 + 12x - 5$ and the domain is the set of all x such that $0 \leq x \leq 9$, then the absolute maximum value of the function f occurs when x is

(A) 0 (B) 2 (C) 4 (D) 6 (E) 9

28. If the substitution $\sqrt{x} = \sin y$ is made in the integrand of $\int_0^{1/2} \frac{\sqrt{x}}{\sqrt{1-x}}\,dx$, the resulting integral is

(A) $\int_0^{1/2} \sin^2 y\,dy$ (B) $2\int_0^{1/2} \frac{\sin^2 y}{\cos y}\,dy$ (C) $2\int_0^{\pi/4} \sin^2 y\,dy$

(D) $\int_0^{\pi/4} \sin^2 y\,dy$ (E) $2\int_0^{\pi/6} \sin^2 y\,dy$

---

29. If $y'' = 2y'$ and if $y = y' = e$ when $x = 0$, then when $x = 1$, $y =$

(A) $\frac{e}{2}\left(e^2+1\right)$ (B) e (C) $\frac{e^3}{2}$ (D) $\frac{e}{2}$ (E) $\frac{\left(e^3-e\right)}{2}$

---

30. $\int_1^2 \frac{x-4}{x^2}\,dx$

(A) $-\frac{1}{2}$ (B) $\ln 2 - 2$ (C) $\ln 2$ (D) 2 (E) $\ln 2 + 2$

---

31. If $f(x) = \ln(\ln x)$, then $f'(x) =$

(A) $\frac{1}{x}$ (B) $\frac{1}{\ln x}$ (C) $\frac{\ln x}{x}$ (D) x (E) $\frac{1}{x\ln x}$

---

32. If $y = x^{\ln x}$, then y' is

(A) $\frac{x^{\ln x}\ln x}{x^2}$

(B) $x^{1/x}\ln x$

(C) $\frac{2x^{\ln x}\ln x}{x}$

(D) $\frac{x^{\ln x}\ln x}{x}$

(E) None of the above

33. Suppose that f is an odd function; i.e., $f(-x)=-f(x)$ for all x. Suppose that $f'(x_0)$ exists. Which of the following must necessarily be equal to $f'(-x_0)$?

(A) $f'(x_0)$

(B) $-f'(x_0)$

(C) $\dfrac{1}{f'(x_0)}$

(D) $-\dfrac{1}{f'(x_0)}$

(E) None of the above

---

34. The average (mean) value of $\sqrt{x}$ over the interval $0 \le x \le 2$ is

(A) $\frac{1}{3}\sqrt{2}$ (B) $\frac{1}{2}\sqrt{2}$ (C) $\frac{2}{3}\sqrt{2}$ (D) 1 (E) $\frac{4}{3}\sqrt{2}$

---

35. The region in the first quadrant bounded by the graph of $y = \sec x$, $x = \frac{\pi}{4}$, and the axes is rotated about the x-axis. What is the volume of the solid generated?

(A) $\frac{\pi^2}{4}$ (B) $\pi - 1$ (C) π (D) 2π (E) $\frac{8\pi}{3}$

---

36. $\int_0^1 \frac{x+1}{x^2+2x-3}\,dx$ is

(A) $-\ln\sqrt{3}$ (B) $-\frac{\ln\sqrt{3}}{2}$ (C) $\frac{1-\ln\sqrt{3}}{2}$ (D) $\ln\sqrt{3}$ (E) divergent

---

37. $\lim_{x\to 0} \frac{1-\cos^2(2x)}{x^2} =$

(A) -2 (B) 0 (C) 1 (D) 2 (E) 4

---

38. If $\int_1^2 f(x-c)\,dx = 5$ where c is a constant, then $\int_{1-c}^{2-c} f(x)\,dx =$

(A) $5+c$ (B) 5 (C) $5-c$ (D) $c-5$ (E) -5

39. Let f and g be differentiable functions such that

$f(1)=2$, $f'(1)=3$, $f'(2)=-4$,

$g(1)=2$, $g'(1)=-3$, $g'(2)=5$.

If $h(x)=f\left(g(x)\right)$, then $h'(1)=$

(A) -9 (B) -4 (C) 0 (D) 12 (E) 15

---

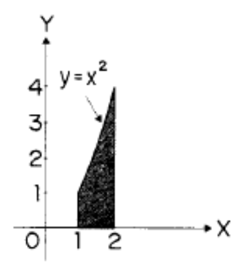
40. The area of the region enclosed by the polar curve $r=1-\cos\theta$ is

(A) $\frac{3}{4}\pi$ (B) π (C) $\frac{3}{2}\pi$ (D) 2π (E) 3π

---

41. Given $f(x)=\begin{cases} x+1 & \text{for } x<0, \\ \cos\pi x & \text{for } x\geq 0, \end{cases}$ $\int_{-1}^{1} f(x)\,dx=$

(A) $\frac{1}{2}+\frac{1}{\pi}$ (B) $-\frac{1}{2}$ (C) $\frac{1}{2}-\frac{1}{\pi}$ (D) $\frac{1}{2}$ (E) $-\frac{1}{2}+\pi$



42. Calculate the approximate area of the shaded region in the figure by the trapezoidal rule, using divisions at $x=\frac{4}{3}$ and $x=\frac{5}{3}$.

(A) $\frac{50}{27}$ (B) $\frac{251}{108}$ (C) $\frac{7}{3}$ (D) $\frac{127}{54}$ (E) $\frac{77}{27}$

43. $\int \arcsin x \, dx =$

(A) $\sin x - \int \frac{x\,dx}{\sqrt{1-x^2}}$

(B) $\frac{(\arcsin x)^2}{2} + C$

(C) $\arcsin x + \int \frac{dx}{\sqrt{1-x^2}}$

(D) $x \arccos x - \int \frac{x\,dx}{\sqrt{1-x^2}}$

(E) $x \arcsin x - \int \frac{x\,dx}{\sqrt{1-x^2}}$

---

44. If f is the solution of $x f'(x) - f(x) = x$ such that $f(-1) = 1$, then $f\left(e^{-1}\right) =$

(A) $-2e^{-1}$ (B) 0 C) e^{-1} (D) $-e^{-1}$ (E) $2e^{-2}$

---

45. Suppose $g'(x) < 0$ for all $x \geq 0$ and $F(x) = \int_0^x t\,g'(t)\,dt$ for all $x \geq 0$. Which of the following statements is FALSE?

(A) F takes on negative values.

(B) F is continuous for all $x > 0$.

(C) $F(x) = x\,g(x) - \int_0^x g(t)\,dt$

(D) $F'(x)$ exists for all $x > 0$.

(E) F is an increasing function.

1973 BC

1. A
2. D
3. A
4. C
5. B
6. D
7. D
8. B
9. A
10. A
11. E
12. D
13. D
14. A
15. C
16. A
17. C
18. D
19. D
20. E
21. B
22. C
23. C
24. A
25. B
26. D
27. E
28. C
29. A
30. B
31. E
32. C
33. A
34. C
35. C
36. E
37. E
38. B
39. D
40. C
41. D
42. D
43. E
44. A
45. E

1985 AP Calculus BC: Section I

90 Minutes—No Calculator

Notes: (1) In this examination, $\ln x$ denotes the natural logarithm of x (that is, logarithm to the base e).

(2) Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

1. The area of the region between the graph of $y = 4x^3 + 2$ and the x-axis from $x = 1$ to $x = 2$ is

(A) 36 (B) 23 (C) 20 (D) 17 (E) 9

---

2. At what values of x does $f(x) = 3x^5 - 5x^3 + 15$ have a relative maximum?

(A) -1 only (B) 0 only (C) 1 only (D) -1 and 1 only (E) -1, 0 and 1

---

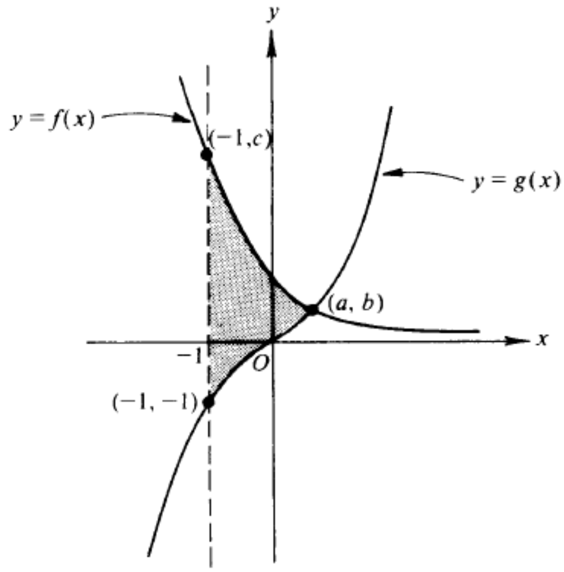
3. $\int_1^2 \frac{x+1}{x^2+2x}\,dx =$

(A) $\ln 8 - \ln 3$ (B) $\frac{\ln 8 - \ln 3}{2}$ (C) $\ln 8$ (D) $\frac{3\ln 2}{2}$ (E) $\frac{3\ln 2 + 2}{2}$

---

4. A particle moves in the xy-plane so that at any time t its coordinates are $x = t^2 - 1$ and $y = t^4 - 2t^3$. At $t = 1$, its acceleration vector is

(A) $(0, -1)$ (B) $(0, 12)$ (C) $(2, -2)$ (D) $(2, 0)$ (E) $(2, 8)$



5. The curves $y = f(x)$ and $y = g(x)$ shown in the figure above intersect at the point (a, b). The area of the shaded region enclosed by these curves and the line $x = -1$ is given by

(A) $\int_0^a (f(x) - g(x))\,dx + \int_{-1}^0 (f(x) + g(x))\,dx$

(B) $\int_{-1}^b g(x)\,dx + \int_b^c f(x)\,dx$

(C) $\int_{-1}^c (f(x) - g(x))\,dx$

(D) $\int_{-1}^a (f(x) - g(x))\,dx$

(E) $\int_{-1}^a (|f(x)| - |g(x)|)\,dx$

---

6. If $f(x) = \dfrac{x}{\tan x}$, then $f'\left(\dfrac{\pi}{4}\right) =$

(A) 2 (B) $\dfrac{1}{2}$ (C) $1 + \dfrac{\pi}{2}$ (D) $\dfrac{\pi}{2} - 1$ (E) $1 - \dfrac{\pi}{2}$

7. Which of the following is equal to $\int \frac{1}{\sqrt{25-x^2}}dx$?

(A) $\arcsin\frac{x}{5}+C$ (B) $\arcsin x+C$ (C) $\frac{1}{5}\arcsin\frac{x}{5}+C$

(D) $\sqrt{25-x^2}+C$ (E) $2\sqrt{25-x^2}+C$

---

8. If f is a function such that $\lim_{x\to 2}\frac{f(x)-f(2)}{x-2}=0$, which of the following must be true?

(A) The limit of $f(x)$ as x approaches 2 does not exist.

(B) f is not defined at $x=2$.

(C) The derivative of f at $x=2$ is 0.

(D) f is continuous at $x=0$.

(E) $f(2)=0$

---

9. If $xy^2+2xy=8$, then, at the point $(1, 2)$, y' is

(A) $-\frac{5}{2}$ (B) $-\frac{4}{3}$ (C) -1 (D) $-\frac{1}{2}$ (E) 0

---

10. For $-1<x<1$ if $f(x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{2n-1}}{2n-1}$, then $f'(x)=$

(A) $\sum_{n=1}^{\infty}(-1)^{n+1}x^{2n-2}$

(B) $\sum_{n=1}^{\infty}(-1)^{n}x^{2n-2}$

(C) $\sum_{n=1}^{\infty}(-1)^{2n}x^{2n}$

(D) $\sum_{n=1}^{\infty}(-1)^{n}x^{2n}$

(E) $\sum_{n=1}^{\infty}(-1)^{n+1}x^{2n}$

11. $\dfrac{d}{dx}\ln\left(\dfrac{1}{1-x}\right) =$

(A) $\dfrac{1}{1-x}$ (B) $\dfrac{1}{x-1}$ (C) $1-x$ (D) $x-1$ (E) $(1-x)^2$

---

12. $\displaystyle\int \frac{dx}{(x-1)(x+2)} =$

(A) $\dfrac{1}{3}\ln\left|\dfrac{x-1}{x+2}\right| + C$ (B) $\dfrac{1}{3}\ln\left|\dfrac{x+2}{x-1}\right| + C$ (C) $\dfrac{1}{3}\ln\left|(x-1)(x+2)\right| + C$

(D) $(\ln|x-1|)(\ln|x+2|) + C$ (E) $\ln\left|(x-1)(x+2)^2\right| + C$

---

13. Let f be the function given by $f(x) = x^3 - 3x^2$. What are all values of c that satisfy the conclusion of the Mean Value Theorem of differential calculus on the closed interval $[0,3]$?

(A) 0 only (B) 2 only (C) 3 only (D) 0 and 3 (E) 2 and 3

---

14. Which of the following series are convergent?

I. $1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots + \dfrac{1}{n^2} + \ldots$

II. $1 + \dfrac{1}{2} + \dfrac{1}{3} + \ldots + \dfrac{1}{n} + \ldots$

III. $1 - \dfrac{1}{3} + \dfrac{1}{3^2} - \ldots + \dfrac{(-1)^{n+1}}{3^{n-1}} + \ldots$

(A) I only (B) III only (C) I and III only (D) II and III only (E) I, II, and III

---

15. If the velocity of a particle moving along the x-axis is $v(t) = 2t - 4$ and if at $t = 0$ its position is 4, then at any time t its position $x(t)$ is

(A) $t^2 - 4t$ (B) $t^2 - 4t - 4$ (C) $t^2 - 4t + 4$ (D) $2t^2 - 4t$ (E) $2t^2 - 4t + 4$

16. Which of the following functions shows that the statement "If a function is continuous at $x = 0$, then it is differentiable at $x = 0$" is false?

(A) $f(x) = x^{-\frac{4}{3}}$ (B) $f(x) = x^{-\frac{1}{3}}$ (C) $f(x) = x^{\frac{1}{3}}$ (D) $f(x) = x^{\frac{4}{3}}$ (E) $f(x) = x^3$

---

17. If $f(x) = x\ln\left(x^2\right)$, then $f'(x) =$

(A) $\ln\left(x^2\right)+1$ (B) $\ln\left(x^2\right)+2$ (C) $\ln\left(x^2\right)+\frac{1}{x}$ (D) $\frac{1}{x^2}$ (E) $\frac{1}{x}$

---

18. $\int \sin(2x+3)\,dx =$

(A) $-2\cos(2x+3)+C$ (B) $-\cos(2x+3)+C$ (C) $-\frac{1}{2}\cos(2x+3)+C$

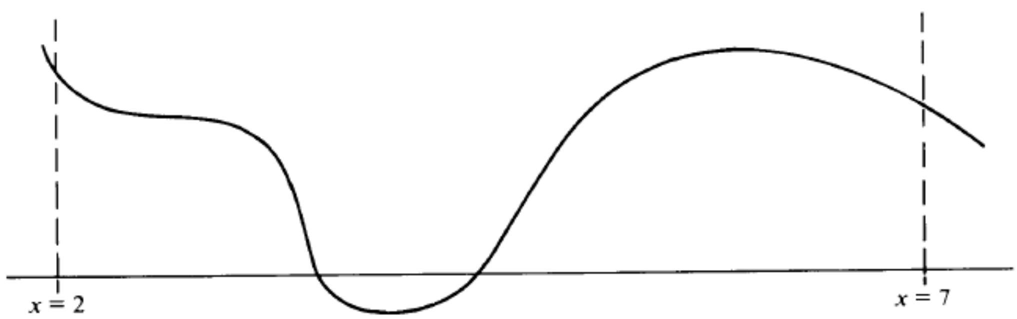
(D) $\frac{1}{2}\cos(2x+3)+C$ (E) $\cos(2x+3)+C$

---

19. If f and g are twice differentiable functions such that $g(x) = e^{f(x)}$ and $g''(x) = h(x)e^{f(x)}$, then $h(x) =$

(A) $f'(x)+f''(x)$ (B) $f'(x)+\left(f''(x)\right)^2$ (C) $\left(f'(x)+f''(x)\right)^2$

(D) $\left(f'(x)\right)^2+f''(x)$ (E) $2f'(x)+f''(x)$



20. The graph of $y = f(x)$ on the closed interval $[2, 7]$ is shown above. How many points of inflection does this graph have on this interval?

(A) One (B) Two (C) Three (D) Four (E) Five

21. If $\int f(x)\sin x\,dx = -f(x)\cos x + \int 3x^2\cos x\,dx$, then $f(x)$ could be

(A) $3x^2$ (B) x^3 (C) $-x^3$ (D) $\sin x$ (E) $\cos x$

---

22. The area of a circular region is increasing at a rate of 96π square meters per second. When the area of the region is 64π square meters, how fast, in meters per second, is the radius of the region increasing?

(A) 6 (B) 8 (C) 16 (D) $4\sqrt{3}$ (E) $12\sqrt{3}$

---

23. $\lim\limits_{h\to 0} \dfrac{\int_1^{1+h}\sqrt{x^5+8}\,dx}{h}$ is

(A) 0 (B) 1 (C) 3 (D) $2\sqrt{2}$ (E) nonexistent

---

24. The area of the region enclosed by the polar curve $r=\sin(2\theta)$ for $0\le\theta\le\frac{\pi}{2}$ is

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) $\frac{\pi}{8}$ (E) $\frac{\pi}{4}$

---

25. A particle moves along the *x*-axis so that at any time *t* its position is given by $x(t)=te^{-2t}$. For what values of *t* is the particle at rest?

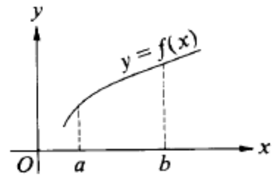
(A) No values (B) 0 only (C) $\frac{1}{2}$ only (D) 1 only (E) 0 and $\frac{1}{2}$

---

26. For $0<x<\frac{\pi}{2}$, if $y=(\sin x)^x$, then $\frac{dy}{dx}$ is

(A) $x\ln(\sin x)$ (B) $(\sin x)^x\cot x$ (C) $x(\sin x)^{x-1}(\cos x)$

(D) $(\sin x)^x(x\cos x+\sin x)$ (E) $(\sin x)^x(x\cot x+\ln(\sin x))$



27. If f is the continuous, strictly increasing function on the interval $a \le x \le b$ as shown above, which of the following must be true?

I. $\int_a^b f(x)\,dx < f(b)(b-a)$

II. $\int_a^b f(x)\,dx > f(a)(b-a)$

III. $\int_a^b f(x)\,dx = f(c)(b-a)$ for some number c such that $a < c < b$

(A) I only (B) II only (C) III only (D) I and III only (E) I, II, and III

---

28. An antiderivative of $f(x) = e^{x+e^x}$ is

(A) $\dfrac{e^{x+e^x}}{1+e^x}$ (B) $\left(1+e^x\right)e^{x+e^x}$ (C) e^{1+e^x} (D) e^{x+e^x} (E) e^{e^x}

---

29. $\displaystyle\lim_{x\to\frac{\pi}{4}} \frac{\sin\left(x-\frac{\pi}{4}\right)}{x-\frac{\pi}{4}}$ is

(A) 0 (B) $\dfrac{1}{\sqrt{2}}$ (C) $\dfrac{\pi}{4}$ (D) 1 (E) nonexistent

---

30. If $x = t^3 - t$ and $y = \sqrt{3t+1}$, then $\dfrac{dy}{dx}$ at $t = 1$ is

(A) $\dfrac{1}{8}$ (B) $\dfrac{3}{8}$ (C) $\dfrac{3}{4}$ (D) $\dfrac{8}{3}$ (E) 8

---

31. What are all values of x for which the series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$ converges?

(A) $-1 \le x < 1$ (B) $-1 \le x \le 1$ (C) $0 < x < 2$ (D) $0 \le x < 2$ (E) $0 \le x \le 2$

32. An equation of the line <u>normal</u> to the graph of $y = x^3 + 3x^2 + 7x - 1$ at the point where $x = -1$ is

(A) $4x + y = -10$ (B) $x - 4y = 23$ (C) $4x - y = 2$ (D) $x + 4y = 25$ (E) $x + 4y = -25$

---

33. If $\dfrac{dy}{dt} = -2y$ and if $y = 1$ when $t = 0$, what is the value of t for which $y = \dfrac{1}{2}$?

(A) $-\dfrac{\ln 2}{2}$ (B) $-\dfrac{1}{4}$ (C) $\dfrac{\ln 2}{2}$ (D) $\dfrac{\sqrt{2}}{2}$ (E) $\ln 2$

---

34. Which of the following gives the area of the surface generated by revolving about the y-axis the arc of $x = y^3$ from $y = 0$ to $y = 1$?

(A) $2\pi\int_0^1 y^3\sqrt{1+9y^4}\,dy$

(B) $2\pi\int_0^1 y^3\sqrt{1+y^6}\,dy$

(C) $2\pi\int_0^1 y^3\sqrt{1+3y^2}\,dy$

(D) $2\pi\int_0^1 y\sqrt{1+9y^4}\,dy$

(E) $2\pi\int_0^1 y\sqrt{1+y^6}\,dy$

---

35. The region in the first quadrant between the x-axis and the graph of $y = 6x - x^2$ is rotated around the y-axis. The volume of the resulting solid of revolution is given by

(A) $\int_0^6 \pi\left(6x - x^2\right)^2 dx$

(B) $\int_0^6 2\pi x\left(6x - x^2\right)dx$

(C) $\int_0^6 \pi x\left(6x - x^2\right)^2 dx$

(D) $\int_0^6 \pi\left(3 + \sqrt{9 - y}\right)^2 dy$

(E) $\int_0^9 \pi\left(3 + \sqrt{9 - y}\right)^2 dy$

36. $\int_{-1}^{1} \frac{3}{x^2}\,dx$ is

(A) -6 (B) -3 (C) 0 (D) 6 (E) nonexistent

---

37. The general solution for the equation $\frac{dy}{dx} + y = xe^{-x}$ is

(A) $y = \frac{x^2}{2}e^{-x} + Ce^{-x}$

(B) $y = \frac{x^2}{2}e^{-x} + e^{-x} + C$

(C) $y = -e^{-x} + \frac{C}{1+x}$

(D) $y = xe^{-x} + Ce^{-x}$

(E) $y = C_1e^{x} + C_2xe^{-x}$

---

38. $\lim_{x\to\infty} \left(1+5e^{x}\right)^{\frac{1}{x}}$ is

(A) 0 (B) 1 (C) e (D) e^5 (E) nonexistent

---

39. The base of a solid is the region enclosed by the graph of $y = e^{-x}$, the coordinate axes, and the line $x = 3$. If all plane cross sections perpendicular to the x-axis are squares, then its volume is

(A) $\frac{\left(1-e^{-6}\right)}{2}$ (B) $\frac{1}{2}e^{-6}$ (C) e^{-6} (D) e^{-3} (E) $1-e^{-3}$

---

40. If the substitution $u = \frac{x}{2}$ is made, the integral $\int_{2}^{4} \frac{1-\left(\frac{x}{2}\right)^2}{x}\,dx =$

(A) $\int_{1}^{2} \frac{1-u^2}{u}\,du$

(B) $\int_{2}^{4} \frac{1-u^2}{u}\,du$

(C) $\int_{1}^{2} \frac{1-u^2}{2u}\,du$

(D) $\int_{1}^{2} \frac{1-u^2}{4u}\,du$

(E) $\int_{2}^{4} \frac{1-u^2}{2u}\,du$

41. What is the length of the arc of $y = \frac{2}{3}x^{\frac{3}{2}}$ from $x = 0$ to $x = 3$?

(A) $\frac{8}{3}$ (B) 4 (C) $\frac{14}{3}$ (D) $\frac{16}{3}$ (E) 7

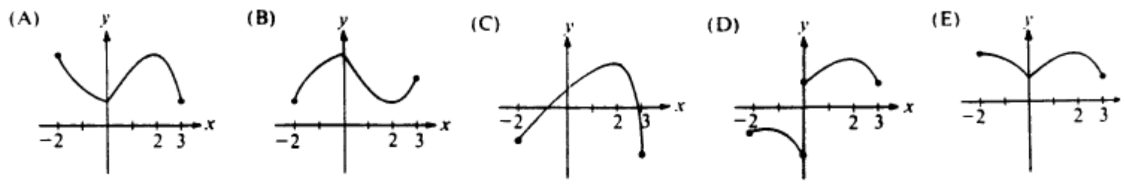
---

42. The coefficient of x^3 in the Taylor series for e^{3x} about $x = 0$ is

(A) $\frac{1}{6}$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{3}{2}$ (E) $\frac{9}{2}$

---

43. Let f be a function that is continuous on the closed interval $[-2,3]$ such that $f'(0)$ does not exist, $f'(2) = 0$, and $f''(x) < 0$ for all x except $x = 0$. Which of the following could be the graph of f?



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44. At each point (x, y) on a certain curve, the slope of the curve is $3x^2y$. If the curve contains the point $(0,8)$, then its equation is

(A) $y = 8e^{x^3}$ (B) $y = x^3 + 8$ (C) $y = e^{x^3} + 7$

(D) $y = \ln(x+1) + 8$ (E) $y^2 = x^3 + 8$

---

45. If n is a positive integer, then $\lim_{n\to\infty} \frac{1}{n}\left[\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \ldots + \left(\frac{3n}{n}\right)^2\right]$ can be expressed as

(A) $\int_0^1 \frac{1}{x^2}\,dx$ (B) $3\int_0^1 \left(\frac{1}{x}\right)^2 dx$ (C) $\int_0^3 \left(\frac{1}{x}\right)^2 dx$

(D) $\int_0^3 x^2\,dx$ (E) $3\int_0^3 x^2\,dx$

1985 BC

1. D
2. A
3. B
4. D
5. D
6. E
7. A
8. C
9. B
10. A
11. A
12. A
13. B
14. C
15. C
16. C
17. B
18. C
19. D
20. C
21. B
22. A
23. C
24. D
25. C
26. E
27. E
28. E
29. D
30. B
31. D
32. E
33. C
34. A
35. B
36. E
37. A
38. C
39. A
40. A
41. C
42. E
43. E
44. A
45. D

90 Minutes—No Calculator

Notes: (1) In this examination, $\ln x$ denotes the natural logarithm of x (that is, logarithm to the base e).

(2) Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

1. The area of the region in the first quadrant enclosed by the graph of $y = x(1-x)$ and the x-axis is

(A) $\frac{1}{6}$ (B) $\frac{1}{3}$ (C) $\frac{2}{3}$ (D) $\frac{5}{6}$ (E) 1

---

2. $\int_0^1 x(x^2+2)^2\,dx =$

(A) $\frac{19}{2}$ (B) $\frac{19}{3}$ (C) $\frac{9}{2}$ (D) $\frac{19}{6}$ (E) $\frac{1}{6}$

---

3. If $f(x) = \ln(\sqrt{x})$, then $f''(x) =$

(A) $-\frac{2}{x^2}$ (B) $-\frac{1}{2x^2}$ (C) $-\frac{1}{2x}$ (D) $-\frac{1}{2x^{\frac{3}{2}}}$ (E) $\frac{2}{x^2}$

---

4. If u, v, and w are nonzero differentiable functions, then the derivative of $\frac{uv}{w}$ is

(A) $\frac{uv' + u'v}{w'}$ (B) $\frac{u'v'w - uvw'}{w^2}$ (C) $\frac{uvw' - uv'w - u'vw}{w^2}$

(D) $\frac{u'vw + uv'w + uvw'}{w^2}$ (E) $\frac{uv'w + u'vw - uvw'}{w^2}$

5. Let f be the function defined by the following.

$$f(x)=\begin{cases}\sin x, & x<0\\ x^2, & 0\le x<1\\ 2-x, & 1\le x<2\\ x-3, & x\ge 2\end{cases}$$

For what values of x is f NOT continuous?

(A) 0 only (B) 1 only (C) 2 only (D) 0 and 2 only (E) 0, 1, and 2

---

6. If $y^2-2xy=16$, then $\frac{dy}{dx}=$

(A) $\frac{x}{y-x}$ (B) $\frac{y}{x-y}$ (C) $\frac{y}{y-x}$ (D) $\frac{y}{2y-x}$ (E) $\frac{2y}{x-y}$

---

7. $\int_2^{+\infty}\frac{dx}{x^2}$ is

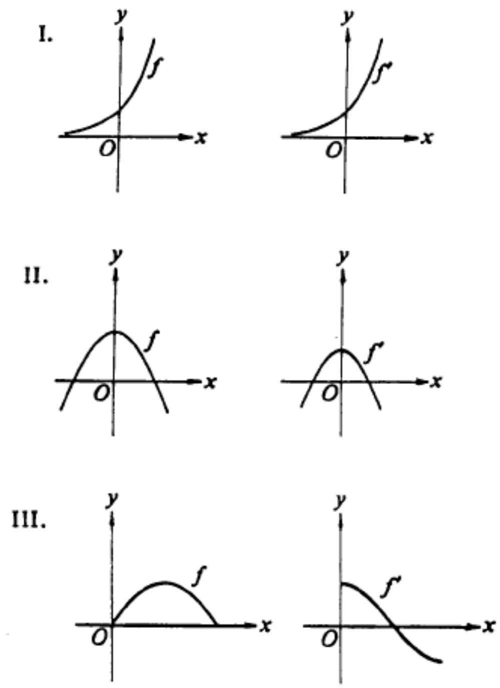
(A) $\frac{1}{2}$ (B) ln 2 (C) 1 (D) 2 (E) nonexistent

---

8. If $f(x)=e^x$, then $\ln\left(f'(2)\right)=$

(A) 2 (B) 0 (C) $\frac{1}{e^2}$ (D) $2e$ (E) e^2

9. Which of the following pairs of graphs could represent the graph of a function and the graph of its derivative?



(A) I only (B) II only (C) III only (D) I and III (E) II and III

---

10. $\lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$ is

(A) 0 (B) 1 (C) $\sin x$ (D) $\cos x$ (E) nonexistent

---

11. If $x + 7y = 29$ is an equation of the line normal to the graph of f at the point $(1, 4)$, then $f'(1) =$

(A) 7 (B) $\frac{1}{7}$ (C) $-\frac{1}{7}$ (D) $-\frac{7}{29}$ (E) -7

---

12. A particle travels in a straight line with a constant acceleration of 3 meters per second per second. If the velocity of the particle is 10 meters per second at time 2 seconds, how far does the particle travel during the time interval when its velocity increases from 4 meters per second to 10 meters per second?

(A) 20 m (B) 14 m (C) 7 m (D) 6 m (E) 3 m

13. $\sin(2x) =$

(A) $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!} + \ldots$

(B) $2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \ldots + \frac{(-1)^{n-1}(2x)^{2n-1}}{(2n-1)!} + \ldots$

(C) $-\frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \ldots + \frac{(-1)^n(2x)^{2n}}{(2n)!} + \ldots$

(D) $\frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots + \frac{x^{2n}}{(2n)!} + \ldots$

(E) $2x + \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + \ldots + \frac{(2x)^{2n-1}}{(2n-1)!} + \ldots$

---

14. If $F(x) = \int_1^{x^2} \sqrt{1+t^3}\, dt$, then $F'(x) =$

(A) $2x\sqrt{1+x^6}$

(B) $2x\sqrt{1+x^3}$

(C) $\sqrt{1+x^6}$

(D) $\sqrt{1+x^3}$

(E) $\int_1^{x^2} \frac{3t^2}{2\sqrt{1+t^3}}\, dt$

---

15. For any time $t \geq 0$, if the position of a particle in the xy-plane is given by $x = t^2 + 1$ and $y = \ln(2t+3)$, then the acceleration vector is

(A) $\left(2t, \frac{2}{(2t+3)}\right)$

(B) $\left(2t, \frac{-4}{(2t+3)^2}\right)$

(C) $\left(2, \frac{4}{(2t+3)^2}\right)$

(D) $\left(2, \frac{2}{(2t+3)^2}\right)$

(E) $\left(2, \frac{-4}{(2t+3)^2}\right)$

16. $\int xe^{2x}dx =$

(A) $\frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C$ (B) $\frac{xe^{2x}}{2} - \frac{e^{2x}}{2} + C$ (C) $\frac{xe^{2x}}{2} + \frac{e^{2x}}{4} + C$

(D) $\frac{xe^{2x}}{2} + \frac{e^{2x}}{2} + C$ (E) $\frac{x^2 e^{2x}}{4} + C$

---

17. $\int_2^3 \frac{3}{(x-1)(x+2)} dx =$

(A) $-\frac{33}{20}$ (B) $-\frac{9}{20}$ (C) $\ln\left(\frac{5}{2}\right)$ (D) $\ln\left(\frac{8}{5}\right)$ (E) $\ln\left(\frac{2}{5}\right)$

---

18. If three equal subdivisions of $[-4, 2]$ are used, what is the trapezoidal approximation of $\int_{-4}^{2} \frac{e^{-x}}{2} dx$?

(A) $e^2 + e^0 + e^{-2}$ (B) $e^4 + e^2 + e^0$ (C) $e^4 + 2e^2 + 2e^0 + e^{-2}$

(D) $\frac{1}{2}\left(e^4 + e^2 + e^0 + e^{-2}\right)$ (E) $\frac{1}{2}\left(e^4 + 2e^2 + 2e^0 + e^{-2}\right)$

---

19. A polynomial $p(x)$ has a relative maximum at $(-2, 4)$, a relative minimum at $(1, 1)$, a relative maximum at $(5, 7)$ and no other critical points. How many zeros does $p(x)$ have?

(A) One (B) Two (C) Three (D) Four (E) Five

---

20. The statement "$\lim_{x \to a} f(x) = L$" means that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

(A) if $0 < |x - a| < \varepsilon$, then $|f(x) - L| < \delta$

(B) if $0 < |f(x) - L| < \varepsilon$, then $|x - a| < \delta$

(C) if $|f(x) - L| < \delta$, then $0 < |x - a| < \varepsilon$

(D) $0 < |x - a| < \delta$ and $|f(x) - L| < \varepsilon$

(E) if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$

21. The average value of $\frac{1}{x}$ on the closed interval $[1,3]$ is

(A) $\frac{1}{2}$ (B) $\frac{2}{3}$ (C) $\frac{\ln 2}{2}$ (D) $\frac{\ln 3}{2}$ (E) $\ln 3$

---

22. If $f(x)=\left(x^2+1\right)^x$, then $f'(x)=$

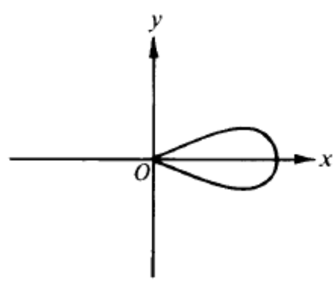
(A) $x\left(x^2+1\right)^{x-1}$

(B) $2x^2\left(x^2+1\right)^{x-1}$

(C) $x\ln\left(x^2+1\right)$

(D) $\ln\left(x^2+1\right)+\frac{2x^2}{x^2+1}$

(E) $\left(x^2+1\right)^x\left[\ln\left(x^2+1\right)+\frac{2x^2}{x^2+1}\right]$



23. Which of the following gives the area of the region enclosed by the loop of the graph of the polar curve $r=4\cos(3\theta)$ shown in the figure above?

(A) $16\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\cos(3\theta)\,d\theta$ (B) $8\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\cos(3\theta)\,d\theta$ (C) $8\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\cos^2(3\theta)\,d\theta$

(D) $16\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\cos^2(3\theta)\,d\theta$ (E) $8\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\cos^2(3\theta)\,d\theta$

24. If c is the number that satisfies the conclusion of the Mean Value Theorem for $f(x) = x^3 - 2x^2$ on the interval $0 \le x \le 2$, then $c =$

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) $\frac{4}{3}$ (E) 2

---

25. The base of a solid is the region in the first quadrant enclosed by the parabola $y = 4x^2$, the line $x = 1$, and the x-axis. Each plane section of the solid perpendicular to the x-axis is a square. The volume of the solid is

(A) $\frac{4\pi}{3}$ (B) $\frac{16\pi}{5}$ (C) $\frac{4}{3}$ (D) $\frac{16}{5}$ (E) $\frac{64}{5}$

---

26. If f is a function such that $f'(x)$ exists for all x and $f(x) > 0$ for all x, which of the following is NOT necessarily true?

(A) $\int_{-1}^{1} f(x)\,dx > 0$

(B) $\int_{-1}^{1} 2f(x)\,dx = 2\int_{-1}^{1} f(x)\,dx$

(C) $\int_{-1}^{1} f(x)\,dx = 2\int_{0}^{1} f(x)\,dx$

(D) $\int_{-1}^{1} f(x)\,dx = -\int_{1}^{-1} f(x)\,dx$

(E) $\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx$

---

27. If the graph of $y = x^3 + ax^2 + bx - 4$ has a point of inflection at $(1, -6)$, what is the value of b?

(A) -3 (B) 0 (C) 1 (D) 3

(E) It cannot be determined from the information given.

28. $\frac{d}{dx}\ln\left|\cos\left(\frac{\pi}{x}\right)\right|$ is

(A) $\frac{-\pi}{x^2\cos\left(\frac{\pi}{x}\right)}$ (B) $-\tan\left(\frac{\pi}{x}\right)$ (C) $\frac{1}{\cos\left(\frac{\pi}{x}\right)}$

(D) $\frac{\pi}{x}\tan\left(\frac{\pi}{x}\right)$ (E) $\frac{\pi}{x^2}\tan\left(\frac{\pi}{x}\right)$

---

29. The region R in the first quadrant is enclosed by the lines $x = 0$ and $y = 5$ and the graph of $y = x^2 + 1$. The volume of the solid generated when R is revolved about the y-axis is

(A) 6π (B) 8π (C) $\frac{34\pi}{3}$ (D) 16π (E) $\frac{544\pi}{15}$

---

30. $\sum_{i=n}^{\infty}\left(\frac{1}{3}\right)^i =$

(A) $\frac{3}{2}-\left(\frac{1}{3}\right)^n$ (B) $\frac{3}{2}\left[1-\left(\frac{1}{3}\right)^n\right]$ (C) $\frac{3}{2}\left(\frac{1}{3}\right)^n$

(D) $\frac{2}{3}\left(\frac{1}{3}\right)^n$ (E) $\frac{2}{3}\left(\frac{1}{3}\right)^{n+1}$

---

31. $\int_0^2 \sqrt{4-x^2}\,dx =$

(A) $\frac{8}{3}$ (B) $\frac{16}{3}$ (C) π (D) 2π (E) 4π

---

32. The general solution of the differential equation $y' = y + x^2$ is $y =$

(A) Ce^x (B) $Ce^x + x^2$ (C) $-x^2 - 2x - 2 + C$

(D) $e^x - x^2 - 2x - 2 + C$ (E) $Ce^x - x^2 - 2x - 2$

33. The length of the curve $y = x^3$ from $x = 0$ to $x = 2$ is given by

(A) $\int_0^2 \sqrt{1+x^6}\, dx$ (B) $\int_0^2 \sqrt{1+3x^2}\, dx$ (C) $\pi\int_0^2 \sqrt{1+9x^4}\, dx$

(D) $2\pi\int_0^2 \sqrt{1+9x^4}\, dx$ (E) $\int_0^2 \sqrt{1+9x^4}\, dx$

---

34. A curve in the plane is defined parametrically by the equations $x = t^3 + t$ and $y = t^4 + 2t^2$. An equation of the line tangent to the curve at $t = 1$ is

(A) $y = 2x$ (B) $y = 8x$ (C) $y = 2x - 1$

(D) $y = 4x - 5$ (E) $y = 8x + 13$

---

35. If k is a positive integer, then $\lim_{x\to+\infty} \frac{x^k}{e^x}$ is

(A) 0 (B) 1 (C) e (D) $k!$ (E) nonexistent

---

36. Let R be the region between the graphs of $y = 1$ and $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{2}$. The volume of the solid obtained by revolving R about the x-axis is given by

(A) $2\pi\int_0^{\frac{\pi}{2}} x\sin x\, dx$ (B) $2\pi\int_0^{\frac{\pi}{2}} x\cos x\, dx$ (C) $\pi\int_0^{\frac{\pi}{2}} (1-\sin x)^2\, dx$

(D) $\pi\int_0^{\frac{\pi}{2}} \sin^2 x\, dx$ (E) $\pi\int_0^{\frac{\pi}{2}} \left(1-\sin^2 x\right) dx$

---

37. A person 2 meters tall walks directly away from a streetlight that is 8 meters above the ground. If the person is walking at a constant rate and the person's shadow is lengthening at the rate of $\frac{4}{9}$ meter per second, at what rate, in meters per second, is the person walking?

(A) $\frac{4}{27}$ (B) $\frac{4}{9}$ (C) $\frac{3}{4}$ (D) $\frac{4}{3}$ (E) $\frac{16}{9}$

38. What are all values of x for which the series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges?

(A) $-1 \le x \le 1$ (B) $-1 < x \le 1$ (C) $-1 \le x < 1$

(D) $-1 < x < 1$ (E) All real x

---

39. If $\frac{dy}{dx} = y\sec^2 x$ and $y = 5$ when $x = 0$, then $y =$

(A) $e^{\tan x} + 4$ (B) $e^{\tan x} + 5$ (C) $5e^{\tan x}$

(D) $\tan x + 5$ (E) $\tan x + 5e^x$

---

40. Let f and g be functions that are differentiable everywhere. If g is the inverse function of f and if $g(-2) = 5$ and $f'(5) = -\frac{1}{2}$, then $g'(-2) =$

(A) 2 (B) $\frac{1}{2}$ (C) $\frac{1}{5}$ (D) $-\frac{1}{5}$ (E) -2

---

41. $\lim_{n\to\infty} \frac{1}{n}\left[\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \ldots + \sqrt{\frac{n}{n}}\right] =$

(A) $\frac{1}{2}\int_0^1 \frac{1}{\sqrt{x}}\,dx$ (B) $\int_0^1 \sqrt{x}\,dx$ (C) $\int_0^1 x\,dx$

(D) $\int_1^2 x\,dx$ (E) $2\int_1^2 x\sqrt{x}\,dx$

---

42. If $\int_1^4 f(x)\,dx = 6$, what is the value of $\int_1^4 f(5-x)\,dx$?

(A) 6 (B) 3 (C) 0 (D) -1 (E) -6

43. Bacteria in a certain culture increase at a rate proportional to the number present. If the number of bacteria doubles in three hours, in how many hours will the number of bacteria triple?

(A) $\frac{3\ln 3}{\ln 2}$ (B) $\frac{2\ln 3}{\ln 2}$ (C) $\frac{\ln 3}{\ln 2}$ (D) $\ln\left(\frac{27}{2}\right)$ (E) $\ln\left(\frac{9}{2}\right)$

---

44. Which of the following series converge?

I. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2n+1}$

II. $\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{3}{2}\right)^n$

III. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

(A) I only
(B) II only
(C) III only
(D) I and III only
(E) I, II, and III

---

45. What is the area of the largest rectangle that can be inscribed in the ellipse $4x^2 + 9y^2 = 36$?

(A) $6\sqrt{2}$ (B) 12 (C) 24 (D) $24\sqrt{2}$ (E) 36

1988 BC

1. A
2. D
3. B
4. E
5. C
6. C
7. A
8. A
9. D
10. D
11. A
12. B
13. B
14. A
15. E
16. A
17. D
18. E
19. B
20. E
21. D
22. E
23. E
24. D
25. D
26. C
27. B
28. E
29. B
30. C
31. C
32. E
33. E
34. C
35. A
36. E or D
37. D
38. C
39. C
40. E
41. B
42. A
43. A
44. A
45. B

90 Minutes—Scientific Calculator

Notes: (1) The exact numerical value of the correct answer does not always appear among the choices given. When this happens, select from among the choices the number that best approximates the exact numerical value.

(2) Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

1. The area of the region enclosed by the graphs of $y = x^2$ and $y = x$ is

(A) $\frac{1}{6}$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{5}{6}$ (E) 1

---

2. If $f(x) = 2x^2 + 1$, then $\lim_{x \to 0} \frac{f(x) - f(0)}{x^2}$ is

(A) 0 (B) 1 (C) 2 (D) 4 (E) nonexistent

---

3. If p is a polynomial of degree n, $n > 0$, what is the degree of the polynomial $Q(x) = \int_0^x p(t)dt$?

(A) 0 (B) 1 (C) $n-1$ (D) n (E) $n+1$

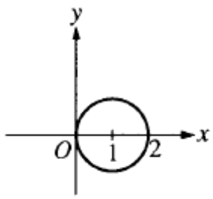
---

4. A particle moves along the curve $xy = 10$. If $x = 2$ and $\frac{dy}{dt} = 3$, what is the value of $\frac{dx}{dt}$?

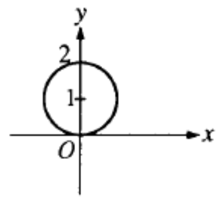
(A) $-\frac{5}{2}$ (B) $-\frac{6}{5}$ (C) 0 (D) $\frac{4}{5}$ (E) $\frac{6}{5}$

5. Which of the following represents the graph of the polar curve $r = 2\sec\theta$?

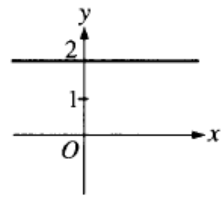
(A)



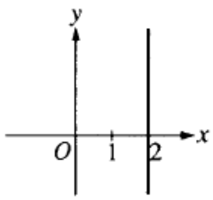
(B)



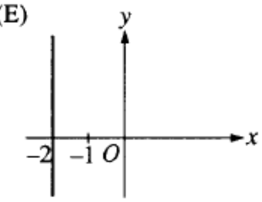
(C)



(D)



(E)



---

6. If $x = t^2 + 1$ and $y = t^3$, then $\dfrac{d^2y}{dx^2} =$

(A) $\dfrac{3}{4t}$ (B) $\dfrac{3}{2t}$ (C) $3t$ (D) $6t$ (E) $\dfrac{3}{2}$

---

7. $\int_0^1 x^3 e^{x^4}\, dx =$

(A) $\dfrac{1}{4}(e-1)$ (B) $\dfrac{1}{4}e$ (C) $e-1$ (D) e (E) $4(e-1)$

---

8. If $f(x) = \ln(e^{2x})$, then $f'(x) =$

(A) 1 (B) 2 (C) $2x$ (D) e^{-2x} (E) $2e^{-2x}$

9. If $f(x) = 1 + x^{\frac{2}{3}}$, which of the following is NOT true?

(A) f is continuous for all real numbers.

(B) f has a minimum at $x = 0$.

(C) f is increasing for $x > 0$.

(D) $f'(x)$ exists for all x.

(E) $f''(x)$ is negative for $x > 0$.

---

10. Which of the following functions are continuous at $x = 1$?

I. $\ln x$

II. e^x

III. $\ln(e^x - 1)$

(A) I only (B) II only (C) I and II only (D) II and III only (E) I, II, and III

---

11. $\int_4^{\infty} \frac{-2x}{\sqrt[3]{9-x^2}}\,dx$ is

(A) $7^{\frac{2}{3}}$ (B) $\frac{3}{2}\left(7^{\frac{2}{3}}\right)$ (C) $9^{\frac{2}{3}} + 7^{\frac{2}{3}}$ (D) $\frac{3}{2}\left(9^{\frac{2}{3}} + 7^{\frac{2}{3}}\right)$ (E) nonexistent

---

12. The position of a particle moving along the x-axis is $x(t) = \sin(2t) - \cos(3t)$ for time $t \geq 0$. When $t = \pi$, the acceleration of the particle is

(A) 9 (B) $\frac{1}{9}$ (C) 0 (D) $-\frac{1}{9}$ (E) -9

---

13. If $\frac{dy}{dx} = x^2 y$, then y could be

(A) $3\ln\left(\frac{x}{3}\right)$ (B) $e^{\frac{x^3}{3}} + 7$ (C) $2e^{\frac{x^3}{3}}$ (D) $3e^{2x}$ (E) $\frac{x^3}{3} + 1$

14. The <u>derivative</u> of f is $x^4(x-2)(x+3)$. At how many points will the graph of f have a relative maximum?

(A) None (B) One (C) Two (D) Three (E) Four

---

15. If $f(x) = e^{\tan^2 x}$, then $f'(x) =$

(A) $e^{\tan^2 x}$

(B) $\sec^2 x\, e^{\tan^2 x}$

(C) $\tan^2 x\, e^{\tan^2 x - 1}$

(D) $2\tan x \sec^2 x\, e^{\tan^2 x}$

(E) $2\tan x\, e^{\tan^2 x}$

---

16. Which of the following series diverge?

I. $\sum_{k=3}^{\infty} \frac{2}{k^2+1}$

II. $\sum_{k=1}^{\infty} \left(\frac{6}{7}\right)^k$

III. $\sum_{k=2}^{\infty} \frac{(-1)^k}{k}$

(A) None (B) II only (C) III only (D) I and III (E) II and III

---

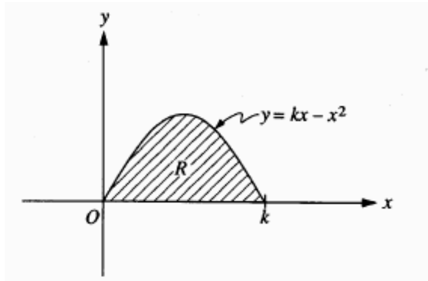
17. The slope of the line tangent to the graph of $\ln(xy) = x$ at the point where $x = 1$ is

(A) 0 (B) 1 (C) e (D) e^2 (E) $1-e$

---

18. If $e^{f(x)} = 1 + x^2$, then $f'(x) =$

(A) $\frac{1}{1+x^2}$ (B) $\frac{2x}{1+x^2}$ (C) $2x(1+x^2)$ (D) $2x\left(e^{1+x^2}\right)$ (E) $2x\ln(1+x^2)$



19. The shaded region R, shown in the figure above, is rotated about the y-axis to form a solid whose volume is 10 cubic units. Of the following, which best approximates k ?

(A) 1.51 (B) 2.09 (C) 2.49 (D) 4.18 (E) 4.77

---

20. A particle moves along the x-axis so that at any time $t \geq 0$ the acceleration of the particle is $a(t) = e^{-2t}$. If at $t = 0$ the velocity of the particle is $\frac{5}{2}$ and its position is $\frac{17}{4}$, then its position at any time $t > 0$ is $x(t) =$

(A) $-\frac{e^{-2t}}{2} + 3$

(B) $\frac{e^{-2t}}{4} + 4$

(C) $4e^{-2t} + \frac{9}{2}t + \frac{1}{4}$

(D) $\frac{e^{-2t}}{2} + 3t + \frac{15}{4}$

(E) $\frac{e^{-2t}}{4} + 3t + 4$

---

21. The value of the derivative of $y = \frac{\sqrt[3]{x^2 + 8}}{\sqrt[4]{2x + 1}}$ at $x = 0$ is

(A) -1 (B) $-\frac{1}{2}$ (C) 0 (D) $\frac{1}{2}$ (E) 1

22. If $f(x) = x^2 e^x$, then the graph of f is decreasing for all x such that

(A) $x < -2$ (B) $-2 < x < 0$ (C) $x > -2$ (D) $x < 0$ (E) $x > 0$

---

23. The length of the curve determined by the equations $x = t^2$ and $y = t$ from $t = 0$ to $t = 4$ is

(A) $\int_0^4 \sqrt{4t+1}\, dt$

(B) $2\int_0^4 \sqrt{t^2+1}\, dt$

(C) $\int_0^4 \sqrt{2t^2+1}\, dt$

(D) $\int_0^4 \sqrt{4t^2+1}\, dt$

(E) $2\pi\int_0^4 \sqrt{4t^2+1}\, dt$

---

24. Let f and g be functions that are differentiable for all real numbers, with $g(x) \neq 0$ for $x \neq 0$. If $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$ and $\lim_{x\to 0} \frac{f'(x)}{g'(x)}$ exists, then $\lim_{x\to 0} \frac{f(x)}{g(x)}$ is

(A) 0

(B) $\frac{f'(x)}{g'(x)}$

(C) $\lim_{x\to 0} \frac{f'(x)}{g'(x)}$

(D) $\frac{f'(x)g(x) - f(x)g'(x)}{(f(x))^2}$

(E) nonexistent

---

25. Consider the curve in the xy-plane represented by $x = e^t$ and $y = te^{-t}$ for $t \geq 0$. The slope of the line tangent to the curve at the point where $x = 3$ is

(A) 20.086 (B) 0.342 (C) -0.005 (D) -0.011 (E) -0.033

26. If $y = \arctan(e^{2x})$, then $\dfrac{dy}{dx} =$

(A) $\dfrac{2e^{2x}}{\sqrt{1-e^{4x}}}$ (B) $\dfrac{2e^{2x}}{1+e^{4x}}$ (C) $\dfrac{e^{2x}}{1+e^{4x}}$ (D) $\dfrac{1}{\sqrt{1-e^{4x}}}$ (E) $\dfrac{1}{1+e^{4x}}$

---

27. The interval of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{(x-1)^n}{3^n}$ is

(A) $-3 < x \le 3$ (B) $-3 \le x \le 3$ (C) $-2 < x < 4$

(D) $-2 \le x < 4$ (E) $0 \le x \le 2$

---

28. If a particle moves in the xy-plane so that at time $t > 0$ its position vector is $\left(\ln(t^2 + 2t), 2t^2\right)$, then at time $t = 2$, its velocity vector is

(A) $\left(\dfrac{3}{4}, 8\right)$ (B) $\left(\dfrac{3}{4}, 4\right)$ (C) $\left(\dfrac{1}{8}, 8\right)$ (D) $\left(\dfrac{1}{8}, 4\right)$ (E) $\left(-\dfrac{5}{16}, 4\right)$

---

29. $\displaystyle\int x \sec^2 x \, dx =$

(A) $x \tan x + C$ (B) $\dfrac{x^2}{2} \tan x + C$ (C) $\sec^2 x + 2\sec^2 x \tan x + C$

(D) $x \tan x - \ln|\cos x| + C$ (E) $x \tan x + \ln|\cos x| + C$

---

30. What is the volume of the solid generated by rotating about the x-axis the region enclosed by the curve $y = \sec x$ and the lines $x = 0$, $y = 0$, and $x = \dfrac{\pi}{3}$?

(A) $\dfrac{\pi}{\sqrt{3}}$

(B) π

(C) $\pi\sqrt{3}$

(D) $\dfrac{8\pi}{3}$

(E) $\pi \ln\left(\dfrac{1}{2} + \sqrt{3}\right)$

31. If $s_n = \left(\frac{(5+n)^{100}}{5^{n+1}}\right)\left(\frac{5^n}{(4+n)^{100}}\right)$, to what number does the sequence $\{s_n\}$ converge?

(A) $\frac{1}{5}$ (B) 1 (C) $\frac{5}{4}$ (D) $\left(\frac{5}{4}\right)^{100}$ (E) The sequence does not converge.

---

32. If $\int_a^b f(x)dx = 5$ and $\int_a^b g(x)dx = -1$, which of the following must be true?

I. $f(x) > g(x)$ for $a \le x \le b$

II. $\int_a^b \big(f(x)+g(x)\big)dx = 4$

III. $\int_a^b \big(f(x)g(x)\big)dx = -5$

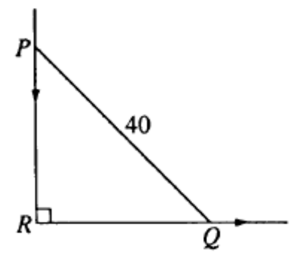
(A) I only (B) II only (C) III only (D) II and III only (E) I, II, and III

---

33. Which of the following is equal to $\int_0^{\pi} \sin x\, dx$?

(A) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\, dx$ (B) $\int_0^{\pi} \cos x\, dx$ (C) $\int_{-\pi}^{0} \sin x\, dx$

(D) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x\, dx$ (E) $\int_{\pi}^{2\pi} \sin x\, dx$



34. In the figure above, PQ represents a 40-foot ladder with end P against a vertical wall and end Q on level ground. If the ladder is slipping down the wall, what is the distance RQ at the instant when Q is moving along the ground $\frac{3}{4}$ as fast as P is moving down the wall?

(A) $\frac{6}{5}\sqrt{10}$ (B) $\frac{8}{5}\sqrt{10}$ (C) $\frac{80}{\sqrt{7}}$ (D) 24 (E) 32

---

35. If F and f are differentiable functions such that $F(x)=\int_0^x f(t)dt$, and if $F(a)=-2$ and $F(b)=-2$ where $a<b$, which of the following must be true?

(A) $f(x)=0$ for some x such that $a<x<b$.

(B) $f(x)>0$ for all x such that $a<x<b$.

(C) $f(x)<0$ for all x such that $a<x<b$.

(D) $F(x)\le 0$ for all x such that $a<x<b$.

(E) $F(x)=0$ for some x such that $a<x<b$.

---

36. Consider all right circular cylinders for which the sum of the height and circumference is 30 centimeters. What is the radius of the one with maximum volume?

(A) 3 cm (B) 10 cm (C) 20 cm (D) $\frac{30}{\pi^2}$ cm (E) $\frac{10}{\pi}$ cm

37. If $f(x)=\begin{cases} x & \text{for } x \le 1 \\ \dfrac{1}{x} & \text{for } x > 1, \end{cases}$ then $\int_0^e f(x)dx =$

(A) 0 (B) $\frac{3}{2}$ (C) 2 (D) e (E) $e+\frac{1}{2}$

---

38. During a certain epidemic, the number of people that are infected at any time increases at a rate proportional to the number of people that are infected at that time. If 1,000 people are infected when the epidemic is first discovered, and 1,200 are infected 7 days later, how many people are infected 12 days after the epidemic is first discovered?

(A) 343 (B) 1,343 (C) 1,367 (D) 1,400 (E) 2,057

---

39. If $\frac{dy}{dx}=\frac{1}{x}$, then the average rate of change of y with respect to x on the closed interval $[1,4]$ is

(A) $-\frac{1}{4}$ (B) $\frac{1}{2}\ln 2$ (C) $\frac{2}{3}\ln 2$ (D) $\frac{2}{5}$ (E) 2

---

40. Let R be the region in the first quadrant enclosed by the x-axis and the graph of $y=\ln(1+2x-x^2)$. If Simpson's Rule with 2 subintervals is used to approximate the area of R, the approximation is

(A) 0.462 (B) 0.693 (C) 0.924 (D) 0.986 (E) 1.850

---

41. Let $f(x)=\int_{-2}^{x^2-3x} e^{t^2}dt$. At what value of x is $f(x)$ a minimum?

(A) For no value of x (B) $\frac{1}{2}$ (C) $\frac{3}{2}$ (D) 2 (E) 3

---

42. $\lim_{x\to 0}(1+2x)^{\csc x} =$

(A) 0 (B) 1 (C) 2 (D) e (E) e^2

43. The coefficient of x^6 in the Taylor series expansion about $x = 0$ for $f(x) = \sin\left(x^2\right)$ is

(A) $-\frac{1}{6}$ (B) 0 (C) $\frac{1}{120}$ (D) $\frac{1}{6}$ (E) 1

---

44. If f is continuous on the interval $[a, b]$, then there exists c such that $a < c < b$ and $\int_a^b f(x)dx =$

(A) $\frac{f(c)}{b-a}$ (B) $\frac{f(b)-f(a)}{b-a}$ (C) $f(b)-f(a)$ (D) $f'(c)(b-a)$ (E) $f(c)(b-a)$

---

45. If $f(x) = \sum_{k=1}^{\infty}\left(\sin^2 x\right)^k$, then $f(1)$ is

(A) 0.369 (B) 0.585 (C) 2.400 (D) 2.426 (E) 3.426

1993 Answer Key

1993 BC

1. A
2. C
3. E
4. B
5. D
6. A
7. A
8. B
9. D
10. E
11. E
12. E
13. C
14. B
15. D
16. A
17. A
18. B
19. B
20. E
21. A
22. B
23. D
24. C
25. D
26. B
27. C
28. A
29. E
30. C
31. A
32. B
33. A
34. E
35. A
36. E
37. B
38. C
39. C
40. C
41. C
42. E
43. A
44. E
45. D

1997 AP Calculus BC: Section I, Part A

50 Minutes—No Calculator

Note: Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

1. $\int_0^1 \sqrt{x}\,(x+1)\,dx =$

(A) 0 (B) 1 (C) $\frac{16}{15}$ (D) $\frac{7}{5}$ (E) 2

---

2. If $x = e^{2t}$ and $y = \sin(2t)$, then $\frac{dy}{dx} =$

(A) $4e^{2t}\cos(2t)$ (B) $\frac{e^{2t}}{\cos(2t)}$ (C) $\frac{\sin(2t)}{2e^{2t}}$ (D) $\frac{\cos(2t)}{2e^{2t}}$ (E) $\frac{\cos(2t)}{e^{2t}}$

---

3. The function f given by $f(x) = 3x^5 - 4x^3 - 3x$ has a relative maximum at $x =$

(A) -1 (B) $-\frac{\sqrt{5}}{5}$ (C) 0 (D) $\frac{\sqrt{5}}{5}$ (E) 1

---

4. $\frac{d}{dx}\left(xe^{\ln x^2}\right) =$

(A) $1+2x$ (B) $x+x^2$ (C) $3x^2$ (D) x^3 (E) x^2+x^3

---

5. If $f(x) = (x-1)^{\frac{3}{2}} + \frac{e^{x-2}}{2}$, then $f'(2) =$

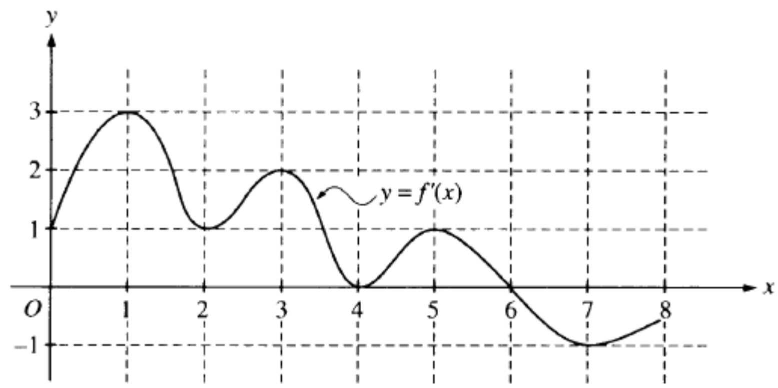
(A) 1 (B) $\frac{3}{2}$ (C) 2 (D) $\frac{7}{2}$ (E) $\frac{3+e}{2}$

---

6. The line normal to the curve $y = \sqrt{16-x}$ at the point $(0, 4)$ has slope

(A) 8 (B) 4 (C) $\frac{1}{8}$ (D) $-\frac{1}{8}$ (E) -8

<u>Questions 7-9</u> refer to the graph and the information below.



The function f is defined on the closed interval $[0,8]$. The graph of its derivative f' is shown above.

7. The point $(3,5)$ is on the graph of $y = f(x)$. An equation of the line tangent to the graph of f at $(3,5)$ is

(A) $y = 2$

(B) $y = 5$

(C) $y - 5 = 2(x - 3)$

(D) $y + 5 = 2(x - 3)$

(E) $y + 5 = 2(x + 3)$

---

8. How many points of inflection does the graph of f have?

(A) Two
(B) Three
(C) Four
(D) Five
(E) Six

9. At what value of x does the absolute minimum of f occur?

(A) 0
(B) 2
(C) 4
(D) 6
(E) 8

---

10. If $y = xy + x^2 + 1$, then when $x = -1$, $\frac{dy}{dx}$ is

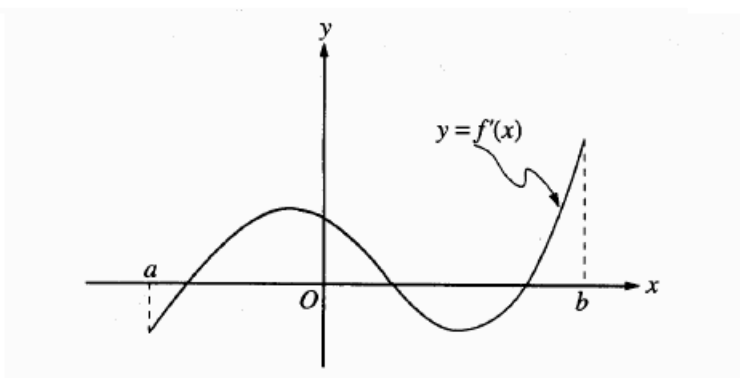
(A) $\frac{1}{2}$ (B) $-\frac{1}{2}$ (C) -1 (D) -2 (E) nonexistent

---

11. $\int_1^\infty \frac{x}{(1+x^2)^2}\,dx$ is

(A) $-\frac{1}{2}$ (B) $-\frac{1}{4}$ (C) $\frac{1}{4}$ (D) $\frac{1}{2}$ (E) divergent

---



12. The graph of f', the derivative of f, is shown in the figure above. Which of the following describes all relative extrema of f on the open interval (a, b)?

(A) One relative maximum and two relative minima
(B) Two relative maxima and one relative minimum
(C) Three relative maxima and one relative minimum
(D) One relative maximum and three relative minima
(E) Three relative maxima and two relative minima

13. A particle moves along the x-axis so that its acceleration at any time t is $a(t) = 2t - 7$. If the initial velocity of the particle is 6, at what time t during the interval $0 \le t \le 4$ is the particle farthest to the right?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

---

14. The sum of the infinite geometric series $\frac{3}{2} + \frac{9}{16} + \frac{27}{128} + \frac{81}{1{,}024} + \dots$ is

(A) 1.60 (B) 2.35 (C) 2.40 (D) 2.45 (E) 2.50

---

15. The length of the path described by the parametric equations $x = \cos^3 t$ and $y = \sin^3 t$, for $0 \le t \le \frac{\pi}{2}$, is given by

(A) $\int_0^{\frac{\pi}{2}} \sqrt{3\cos^2 t + 3\sin^2 t}\, dt$

(B) $\int_0^{\frac{\pi}{2}} \sqrt{-3\cos^2 t \sin t + 3\sin^2 t \cos t}\, dt$

(C) $\int_0^{\frac{\pi}{2}} \sqrt{9\cos^4 t + 9\sin^4 t}\, dt$

(D) $\int_0^{\frac{\pi}{2}} \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t}\, dt$

(E) $\int_0^{\frac{\pi}{2}} \sqrt{\cos^6 t + \sin^6 t}\, dt$

---

16. $\lim\limits_{h \to 0} \frac{e^h - 1}{2h}$ is

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) e (E) nonexistent

17. Let f be the function given by $f(x) = \ln(3-x)$. The third-degree Taylor polynomial for f about $x = 2$ is

(A) $-(x-2)+\frac{(x-2)^2}{2}-\frac{(x-2)^3}{3}$

(B) $-(x-2)-\frac{(x-2)^2}{2}-\frac{(x-2)^3}{3}$

(C) $(x-2)+(x-2)^2+(x-2)^3$

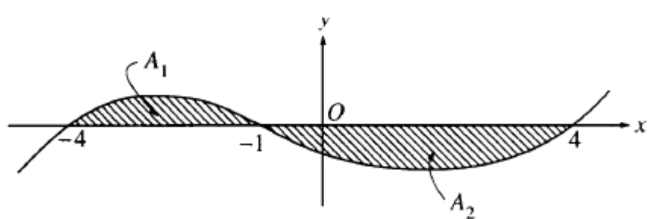
(D) $(x-2)+\frac{(x-2)^2}{2}+\frac{(x-2)^3}{3}$

(E) $(x-2)-\frac{(x-2)^2}{2}+\frac{(x-2)^3}{3}$

---

18. For what values of t does the curve given by the parametric equations $x = t^3 - t^2 - 1$ and $y = t^4 + 2t^2 - 8t$ have a vertical tangent?

(A) 0 only
(B) 1 only
(C) 0 and $\frac{2}{3}$ only
(D) 0, $\frac{2}{3}$, and 1
(E) No value



19. The graph of $y = f(x)$ is shown in the figure above. If A_1 and A_2 are positive numbers that represent the areas of the shaded regions, then in terms of A_1 and A_2,

$\int_{-4}^{4} f(x)\,dx - 2\int_{-1}^{4} f(x)\,dx =$

(A) A_1 (B) $A_1 - A_2$ (C) $2A_1 - A_2$ (D) $A_1 + A_2$ (E) $A_1 + 2A_2$

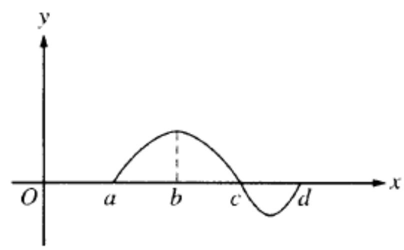
20. What are all values of x for which the series $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n \cdot 3^n}$ converges?

(A) $-3 \le x \le 3$
(B) $-3 < x < 3$
(C) $-1 < x \le 5$
(D) $-1 \le x \le 5$
(E) $-1 \le x < 5$

---

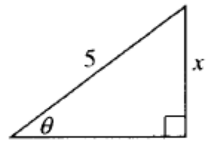
21. Which of the following is equal to the area of the region inside the polar curve $r = 2\cos\theta$ and outside the polar curve $r = \cos\theta$?

(A) $3\int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta$ (B) $3\int_0^{\pi} \cos^2\theta\, d\theta$ (C) $\frac{3}{2}\int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta$ (D) $3\int_0^{\frac{\pi}{2}} \cos\theta\, d\theta$ (E) $3\int_0^{\pi} \cos\theta\, d\theta$



22. The graph of f is shown in the figure above. If $g(x) = \int_a^x f(t)\,dt$, for what value of x does $g(x)$ have a maximum?

(A) a
(B) b
(C) c
(D) d
(E) It cannot be determined from the information given.



23. In the triangle shown above, if θ increases at a constant rate of 3 radians per minute, at what rate is x increasing in units per minute when x equals 3 units?

(A) 3 (B) $\frac{15}{4}$ (C) 4 (D) 9 (E) 12

---

24. The Taylor series for $\sin x$ about $x = 0$ is $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$. If f is a function such that $f'(x) = \sin\left(x^2\right)$, then the coefficient of x^7 in the Taylor series for $f(x)$ about $x = 0$ is

(A) $\frac{1}{7!}$ (B) $\frac{1}{7}$ (C) 0 (D) $-\frac{1}{42}$ (E) $-\frac{1}{7!}$

---

25. The closed interval $[a, b]$ is partitioned into n equal subintervals, each of width Δx, by the numbers $x_0, x_1, \dots, x_n$ where $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$. What is $\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{x_i}\, \Delta x$?

(A) $\frac{2}{3}\left(b^{\frac{3}{2}} - a^{\frac{3}{2}}\right)$

(B) $b^{\frac{3}{2}} - a^{\frac{3}{2}}$

(C) $\frac{3}{2}\left(b^{\frac{3}{2}} - a^{\frac{3}{2}}\right)$

(D) $b^{\frac{1}{2}} - a^{\frac{1}{2}}$

(E) $2\left(b^{\frac{1}{2}} - a^{\frac{1}{2}}\right)$

1997 AP Calculus BC: Section I, Part B

40 Minutes—Graphing Calculator Required

Notes: (1) The exact numerical value of the correct answer does not always appear among the choices given. When this happens, select from among the choices the number that best approximates the exact numerical value.

(2) Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

76. Which of the following sequences converge?

I. $\left\{\frac{5n}{2n-1}\right\}$

II. $\left\{\frac{e^n}{n}\right\}$

III. $\left\{\frac{e^n}{1+e^n}\right\}$

(A) I only (B) II only (C) I and II only (D) I and III only (E) I, II, and III

---

77. When the region enclosed by the graphs of $y = x$ and $y = 4x - x^2$ is revolved about the y-axis, the volume of the solid generated is given by

(A) $\pi\int_0^3 \left(x^3 - 3x^2\right) dx$

(B) $\pi\int_0^3 \left(x^2 - \left(4x - x^2\right)^2\right) dx$

(C) $\pi\int_0^3 \left(3x - x^2\right)^2 dx$

(D) $2\pi\int_0^3 \left(x^3 - 3x^2\right) dx$

(E) $2\pi\int_0^3 \left(3x^2 - x^3\right) dx$

78. $\lim_{h \to 0} \frac{\ln(e+h)-1}{h}$ is

(A) $f'(e)$, where $f(x) = \ln x$

(B) $f'(e)$, where $f(x) = \frac{\ln x}{x}$

(C) $f'(1)$, where $f(x) = \ln x$

(D) $f'(1)$, where $f(x) = \ln(x+e)$

(E) $f'(0)$, where $f(x) = \ln x$

---

79. The position of an object attached to a spring is given by $y(t) = \frac{1}{6}\cos(5t) - \frac{1}{4}\sin(5t)$, where t is time in seconds. In the first 4 seconds, how many times is the velocity of the object equal to 0?

(A) Zero
(B) Three
(C) Five
(D) Six
(E) Seven

---

80. Let f be the function given by $f(x) = \cos(2x) + \ln(3x)$. What is the least value of x at which the graph of f changes concavity?

(A) 0.56 (B) 0.93 (C) 1.18 (D) 2.38 (E) 2.44

---

81. Let f be a continuous function on the closed interval $[-3, 6]$. If $f(-3) = -1$ and $f(6) = 3$, then the Intermediate Value Theorem guarantees that

(A) $f(0) = 0$

(B) $f'(c) = \frac{4}{9}$ for at least one c between -3 and 6

(C) $-1 \le f(x) \le 3$ for all x between -3 and 6

(D) $f(c) = 1$ for at least one c between -3 and 6

(E) $f(c) = 0$ for at least one c between -1 and 3

82. If $0 \le x \le 4$, of the following, which is the greatest value of x such that $\int_0^x (t^2 - 2t)\,dt \ge \int_2^x t\,dt$?

(A) 1.35 (B) 1.38 (C) 1.41 (D) 1.48 (E) 1.59

---

83. If $\frac{dy}{dx} = (1 + \ln x)\,y$ and if $y = 1$ when $x = 1$, then $y =$

(A) $e^{\frac{x^2-1}{x^2}}$

(B) $1 + \ln x$

(C) $\ln x$

(D) $e^{2x + x\ln x - 2}$

(E) $e^{x \ln x}$

---

84. $\int x^2 \sin x\,dx =$

(A) $-x^2 \cos x - 2x \sin x - 2\cos x + C$

(B) $-x^2 \cos x + 2x \sin x - 2\cos x + C$

(C) $-x^2 \cos x + 2x \sin x + 2\cos x + C$

(D) $-\frac{x^3}{3} \cos x + C$

(E) $2x \cos x + C$

---

85. Let f be a twice differentiable function such that $f(1) = 2$ and $f(3) = 7$. Which of the following must be true for the function f on the interval $1 \le x \le 3$?

I. The average rate of change of f is $\frac{5}{2}$.

II. The average value of f is $\frac{9}{2}$.

III. The average value of f' is $\frac{5}{2}$.

(A) None
(B) I only
(C) III only
(D) I and III only
(E) II and III only

86. $\int \frac{dx}{(x-1)(x+3)} =$

(A) $\frac{1}{4}\ln\left|\frac{x-1}{x+3}\right| + C$

(B) $\frac{1}{4}\ln\left|\frac{x+3}{x-1}\right| + C$

(C) $\frac{1}{2}\ln\left|(x-1)(x+3)\right| + C$

(D) $\frac{1}{2}\ln\left|\frac{2x+2}{(x-1)(x+3)}\right| + C$

(E) $\ln\left|(x-1)(x+3)\right| + C$

---

87. The base of a solid is the region in the first quadrant enclosed by the graph of $y = 2 - x^2$ and the coordinate axes. If every cross section of the solid perpendicular to the y-axis is a square, the volume of the solid is given by

(A) $\pi\int_0^2 (2-y)^2\,dy$

(B) $\int_0^2 (2-y)\,dy$

(C) $\pi\int_0^{\sqrt{2}} \left(2-x^2\right)^2 dx$

(D) $\int_0^{\sqrt{2}} \left(2-x^2\right)^2 dx$

(E) $\int_0^{\sqrt{2}} \left(2-x^2\right) dx$

88. Let $f(x) = \int_0^{x^2} \sin t \, dt$. At how many points in the closed interval $\left[0, \sqrt{\pi}\right]$ does the instantaneous rate of change of f equal the average rate of change of f on that interval?

(A) Zero
(B) One
(C) Two
(D) Three
(E) Four

---

89. If f is the antiderivative of $\frac{x^2}{1+x^5}$ such that $f(1) = 0$, then $f(4) =$

(A) -0.012 (B) 0 (C) 0.016 (D) 0.376 (E) 0.629

---

90. A force of 10 pounds is required to stretch a spring 4 inches beyond its natural length. Assuming Hooke's law applies, how much work is done in stretching the spring from its natural length to 6 inches beyond its natural length?

(A) 60.0 inch-pounds
(B) 45.0 inch-pounds
(C) 40.0 inch-pounds
(D) 15.0 inch-pounds
(E) 7.2 inch-pounds

1997 BC

1. C
2. E
3. A
4. C
5. C
6. A
7. C
8. E
9. A
10. B
11. C
12. A
13. B
14. C
15. D
16. B
17. B
18. C
19. D
20. E

21. A
22. C
23. E
24. D
25. A
76. D
77. E
78. A
79. D
80. B
81. D
82. B
83. E
84. C
85. D
86. A
87. B
88. C
89. D
90. B

1998 AP Calculus BC: Section I, Part A

55 Minutes—No Calculator

Note: Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

1. What are all values of x for which the function f defined by $f(x) = x^3 + 3x^2 - 9x + 7$ is increasing?

(A) $-3 < x < 1$
(B) $-1 < x < 1$
(C) $x < -3$ or $x > 1$
(D) $x < -1$ or $x > 3$
(E) All real numbers

---

2. In the xy-plane, the graph of the parametric equations $x = 5t + 2$ and $y = 3t$, for $-3 \le t \le 3$, is a line segment with slope

(A) $\frac{3}{5}$ (B) $\frac{5}{3}$ (C) 3 (D) 5 (E) 13

---

3. The slope of the line tangent to the curve $y^2 + (xy+1)^3 = 0$ at $(2, -1)$ is

(A) $-\frac{3}{2}$ (B) $-\frac{3}{4}$ (C) 0 (D) $\frac{3}{4}$ (E) $\frac{3}{2}$

---

4. $\int \frac{1}{x^2 - 6x + 8}\,dx =$

(A) $\frac{1}{2}\ln\left|\frac{x-4}{x-2}\right| + C$

(B) $\frac{1}{2}\ln\left|\frac{x-2}{x-4}\right| + C$

(C) $\frac{1}{2}\ln\left|(x-2)(x-4)\right| + C$

(D) $\frac{1}{2}\ln\left|(x-4)(x+2)\right| + C$

(E) $\ln\left|(x-2)(x-4)\right| + C$

5. If f and g are twice differentiable and if $h(x) = f\left(g(x)\right)$, then $h''(x) =$

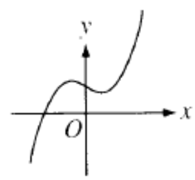
(A) $f''\left(g(x)\right)\left[g'(x)\right]^2 + f'\left(g(x)\right)g''(x)$

(B) $f''\left(g(x)\right)g'(x) + f'\left(g(x)\right)g''(x)$

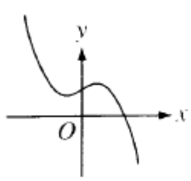
(C) $f''\left(g(x)\right)\left[g'(x)\right]^2$

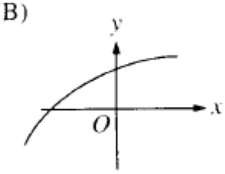
(D) $f''\left(g(x)\right)g''(x)$

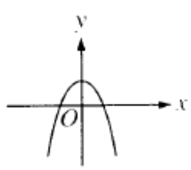
(E) $f''\left(g(x)\right)$

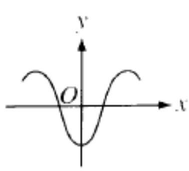


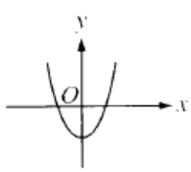
6. The graph of $y = h(x)$ is shown above. Which of the following could be the graph of $y = h'(x)$?

(A)


(B)


(C)


(D)


(E)


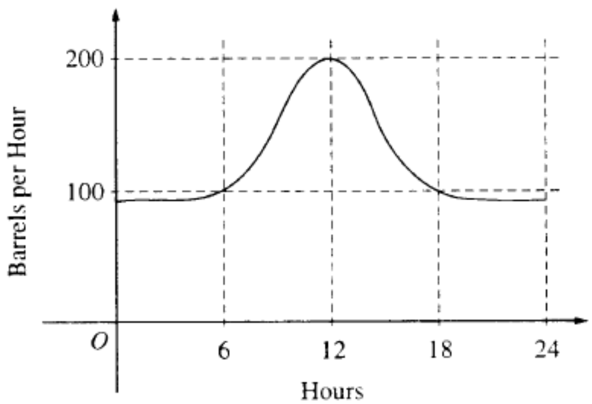
7. $\int_1^e \left(\frac{x^2-1}{x}\right) dx =$

(A) $e-\frac{1}{e}$ (B) e^2-e (C) $\frac{e^2}{2}-e+\frac{1}{2}$ (D) e^2-2 (E) $\frac{e^2}{2}-\frac{3}{2}$

---

8. If $\frac{dy}{dx}=\sin x\cos^2 x$ and if $y=0$ when $x=\frac{\pi}{2}$, what is the value of y when $x=0$?

(A) -1 (B) $-\frac{1}{3}$ (C) 0 (D) $\frac{1}{3}$ (E) 1



9. The flow of oil, in barrels per hour, through a pipeline on July 9 is given by the graph shown above. Of the following, which best approximates the total number of barrels of oil that passed through the pipeline that day?

(A) 500 (B) 600 (C) 2,400 (D) 3,000 (E) 4,800

---

10. A particle moves on a plane curve so that at any time $t>0$ its x-coordinate is t^3-t and its y-coordinate is $(2t-1)^3$. The acceleration vector of the particle at $t=1$ is

(A) $(0,1)$ (B) $(2,3)$ (C) $(2,6)$ (D) $(6,12)$ (E) $(6,24)$

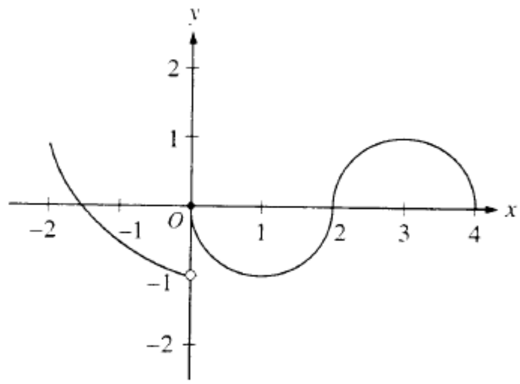
---

11. If f is a linear function and $0<a<b$, then $\int_a^b f''(x)\,dx =$

(A) 0 (B) 1 (C) $\frac{ab}{2}$ (D) $b-a$ (E) $\frac{b^2-a^2}{2}$

12. If $f(x) = \begin{cases} \ln x & \text{for } 0 < x \le 2 \\ x^2 \ln 2 & \text{for } 2 < x \le 4, \end{cases}$ then $\lim_{x \to 2} f(x)$ is

(A) $\ln 2$ (B) $\ln 8$ (C) $\ln 16$ (D) 4 (E) nonexistent



13. The graph of the function f shown in the figure above has a vertical tangent at the point $(2, 0)$ and horizontal tangents at the points $(1, -1)$ and $(3, 1)$. For what values of x, $-2 < x < 4$, is f not differentiable?

(A) 0 only (B) 0 and 2 only (C) 1 and 3 only (D) 0, 1, and 3 only (E) 0, 1, 2, and 3

---

14. What is the approximation of the value of $\sin 1$ obtained by using the fifth-degree Taylor polynomial about $x = 0$ for $\sin x$?

(A) $1 - \frac{1}{2} + \frac{1}{24}$

(B) $1 - \frac{1}{2} + \frac{1}{4}$

(C) $1 - \frac{1}{3} + \frac{1}{5}$

(D) $1 - \frac{1}{4} + \frac{1}{8}$

(E) $1 - \frac{1}{6} + \frac{1}{120}$

15. $\int x \cos x \, dx =$

(A) $x \sin x - \cos x + C$

(B) $x \sin x + \cos x + C$

(C) $-x \sin x + \cos x + C$

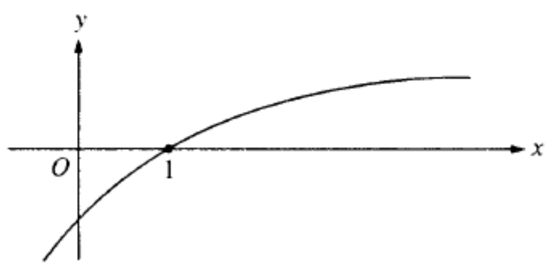
(D) $x \sin x + C$

(E) $\frac{1}{2} x^2 \sin x + C$

---

16. If f is the function defined by $f(x) = 3x^5 - 5x^4$, what are all the x-coordinates of points of inflection for the graph of f?

(A) -1 (B) 0 (C) 1 (D) 0 and 1 (E) -1, 0, and 1



17. The graph of a twice-differentiable function f is shown in the figure above. Which of the following is true?

(A) $f(1) < f'(1) < f''(1)$

(B) $f(1) < f''(1) < f'(1)$

(C) $f'(1) < f(1) < f''(1)$

(D) $f''(1) < f(1) < f'(1)$

(E) $f''(1) < f'(1) < f(1)$

18. Which of the following series converge?

I. $\sum_{n=1}^{\infty} \frac{n}{n+2}$ II. $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n}$ III. $\sum_{n=1}^{\infty} \frac{1}{n}$

(A) None
(B) II only
(C) III only
(D) I and II only
(E) I and III only

---

19. The area of the region inside the polar curve $r = 4\sin\theta$ and outside the polar curve $r = 2$ is given by

(A) $\frac{1}{2}\int_0^{\pi} (4\sin\theta - 2)^2\, d\theta$ (B) $\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (4\sin\theta - 2)^2\, d\theta$ (C) $\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (4\sin\theta - 2)^2\, d\theta$

(D) $\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(16\sin^2\theta - 4\right) d\theta$ (E) $\frac{1}{2}\int_0^{\pi} \left(16\sin^2\theta - 4\right) d\theta$

---

20. When $x = 8$, the rate at which $\sqrt[3]{x}$ is increasing is $\frac{1}{k}$ times the rate at which x is increasing. What is the value of k?

(A) 3 (B) 4 (C) 6 (D) 8 (E) 12

---

21. The length of the path described by the parametric equations $x = \frac{1}{3}t^3$ and $y = \frac{1}{2}t^2$, where $0 \le t \le 1$, is given by

(A) $\int_0^1 \sqrt{t^2 + 1}\, dt$

(B) $\int_0^1 \sqrt{t^2 + t}\, dt$

(C) $\int_0^1 \sqrt{t^4 + t^2}\, dt$

(D) $\frac{1}{2}\int_0^1 \sqrt{4 + t^4}\, dt$

(E) $\frac{1}{6}\int_0^1 t^2\sqrt{4t^2 + 9}\, dt$

22. If $\lim_{b\to\infty}\int_1^b \frac{dx}{x^p}$ is finite, then which of the following must be true?

(A) $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges

(B) $\sum_{n=1}^{\infty}\frac{1}{n^p}$ diverges

(C) $\sum_{n=1}^{\infty}\frac{1}{n^{p-2}}$ converges

(D) $\sum_{n=1}^{\infty}\frac{1}{n^{p-1}}$ converges

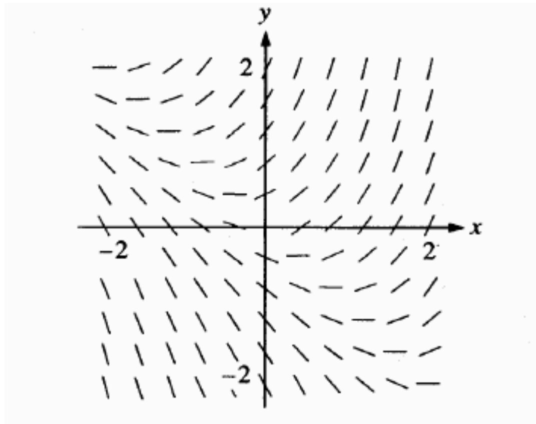
(E) $\sum_{n=1}^{\infty}\frac{1}{n^{p+1}}$ diverges

---

23. Let f be a function defined and continuous on the closed interval $[a,b]$. If f has a relative maximum at c and $a < c < b$, which of the following statements must be true?

I. $f'(c)$ exists.
II. If $f'(c)$ exists, then $f'(c) = 0$.
III. If $f''(c)$ exists, then $f''(c) \le 0$.

(A) II only (B) III only (C) I and II only (D) I and III only (E) II and III only



24. Shown above is a slope field for which of the following differential equations?

(A) $\frac{dy}{dx} = 1 + x$ (B) $\frac{dy}{dx} = x^2$ (C) $\frac{dy}{dx} = x + y$ (D) $\frac{dy}{dx} = \frac{x}{y}$ (E) $\frac{dy}{dx} = \ln y$

---

25. $\int_0^{\infty} x^2 e^{-x^3}\,dx$ is

(A) $-\frac{1}{3}$ (B) 0 (C) $\frac{1}{3}$ (D) 1 (E) divergent

---

26. The population $P(t)$ of a species satisfies the logistic differential equation $\frac{dP}{dt} = P\left(2 - \frac{P}{5000}\right)$, where the initial population $P(0) = 3,000$ and t is the time in years. What is $\lim_{t\to\infty} P(t)$?

(A) 2,500 (B) 3,000 (C) 4,200 (D) 5,000 (E) 10,000

---

27. If $\sum_{n=0}^{\infty} a_n x^n$ is a Taylor series that converges to $f(x)$ for all real x, then $f'(1) =$

(A) 0 (B) a_1 (C) $\sum_{n=0}^{\infty} a_n$ (D) $\sum_{n=1}^{\infty} n a_n$ (E) $\sum_{n=1}^{\infty} n a_n^{\,n-1}$

---

28. $\lim_{x\to 1} \frac{\int_1^x e^{t^2}\,dt}{x^2 - 1}$ is

(A) 0 (B) 1 (C) $\frac{e}{2}$ (D) e (E) nonexistent

50 Minutes—Graphing Calculator Required

Notes: (1) The <u>exact</u> numerical value of the correct answer does not always appear among the choices given. When this happens, select from among the choices the number that <u>best approximates</u> the exact numerical value.

(2) Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

76. For what integer k, $k > 1$, will both $\sum_{n=1}^{\infty} \frac{(-1)^{kn}}{n}$ and $\sum_{n=1}^{\infty} \left(\frac{k}{4}\right)^n$ converge?

(A) 6 (B) 5 (C) 4 (D) 3 (E) 2

---

77. If f is a vector-valued function defined by $f(t) = \left(e^{-t}, \cos t\right)$, then $f''(t) =$

(A) $-e^{-t} + \sin t$

(B) $e^{-t} - \cos t$

(C) $\left(-e^{-t}, -\sin t\right)$

(D) $\left(e^{-t}, \cos t\right)$

(E) $\left(e^{-t}, -\cos t\right)$

---

78. The radius of a circle is decreasing at a constant rate of 0.1 centimeter per second. In terms of the circumference C, what is the rate of change of the area of the circle, in square centimeters per second?

(A) $-(0.2)\pi C$

(B) $-(0.1)C$

(C) $-\frac{(0.1)C}{2\pi}$

(D) $(0.1)^2 C$

(E) $(0.1)^2 \pi C$

79. Let f be the function given by $f(x)=\frac{(x-1)(x^2-4)}{x^2-a}$. For what positive values of a is f continuous for all real numbers x?

(A) None
(B) 1 only
(C) 2 only
(D) 4 only
(E) 1 and 4 only

---

80. Let R be the region enclosed by the graph of $y=1+\ln\left(\cos^4 x\right)$, the x-axis, and the lines $x=-\frac{2}{3}$ and $x=\frac{2}{3}$. The closest integer approximation of the area of R is

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

---

81. If $\frac{dy}{dx}=\sqrt{1-y^2}$, then $\frac{d^2y}{dx^2}=$

(A) $-2y$ (B) $-y$ (C) $\frac{-y}{\sqrt{1-y^2}}$ (D) y (E) $\frac{1}{2}$

---

82. If $f(x)=g(x)+7$ for $3\le x\le 5$, then $\int_3^5 \left[f(x)+g(x)\right]dx=$

(A) $2\int_3^5 g(x)\,dx+7$

(B) $2\int_3^5 g(x)\,dx+14$

(C) $2\int_3^5 g(x)\,dx+28$

(D) $\int_3^5 g(x)\,dx+7$

(E) $\int_3^5 g(x)\,dx+14$

83. The Taylor series for $\ln x$, centered at $x=1$, is $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n}$. Let f be the function given by the sum of the first three nonzero terms of this series. The maximum value of $\left|\ln x - f(x)\right|$ for $0.3 \le x \le 1.7$ is

(A) 0.030 (B) 0.039 (C) 0.145 (D) 0.153 (E) 0.529

---

84. What are all values of x for which the series $\sum_{n=1}^{\infty}\frac{(x+2)^n}{\sqrt{n}}$ converges?

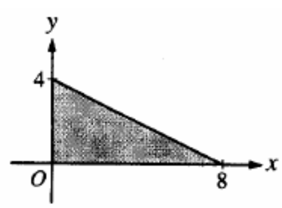
(A) $-3 < x < -1$ (B) $-3 \le x < -1$ (C) $-3 \le x \le -1$ (D) $-1 \le x < 1$ (E) $-1 \le x \le 1$

---

x	2	5	7	8
$f(x)$	10	30	40	20

85. The function f is continuous on the closed interval $[2,8]$ and has values that are given in the table above. Using the subintervals $[2,5]$, $[5,7]$, and $[7,8]$, what is the trapezoidal approximation of $\int_2^8 f(x)\,dx$?

(A) 110 (B) 130 (C) 160 (D) 190 (E) 210

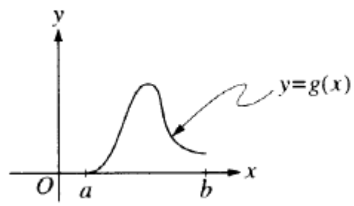


86. The base of a solid is a region in the first quadrant bounded by the x-axis, the y-axis, and the line $x+2y=8$, as shown in the figure above. If cross sections of the solid perpendicular to the x-axis are semicircles, what is the volume of the solid?

(A) 12.566 (B) 14.661 (C) 16.755 (D) 67.021 (E) 134.041

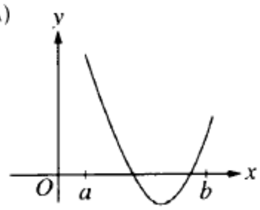
87. Which of the following is an equation of the line tangent to the graph of $f(x)=x^4+2x^2$ at the point where $f'(x)=1$?

(A) $y=8x-5$
(B) $y=x+7$
(C) $y=x+0.763$
(D) $y=x-0.122$
(E) $y=x-2.146$

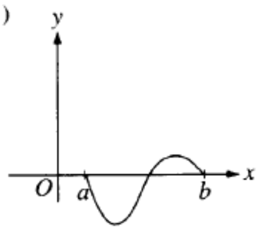


88. Let $g(x)=\int_a^x f(t)\,dt$, where $a \le x \le b$. The figure above shows the graph of g on $[a,b]$. Which of the following could be the graph of f on $[a,b]$?

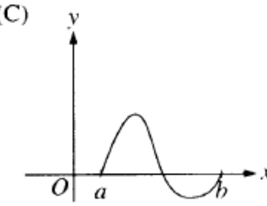
(A)



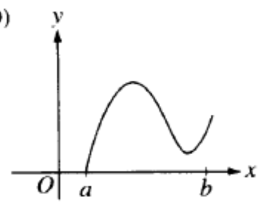
(B)



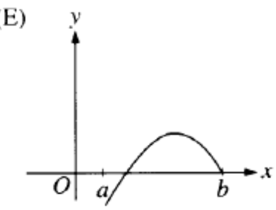
(C)



(D)



(E)



89. The graph of the function represented by the Maclaurin series

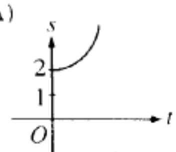
$1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots + \frac{(-1)^n x^n}{n!} + \ldots$ intersects the graph of $y = x^3$ at $x =$

(A) 0.773 (B) 0.865 (C) 0.929 (D) 1.000 (E) 1.857

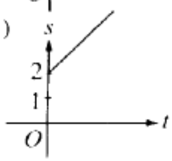
---

90. A particle starts from rest at the point $(2, 0)$ and moves along the x-axis with a constant positive acceleration for time $t \geq 0$. Which of the following could be the graph of the distance $s(t)$ of the particle from the origin as a function of time t?

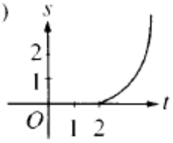
(A)



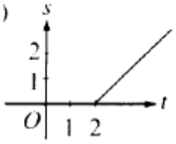
(B)



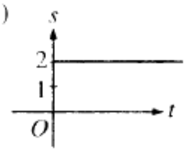
(C)



(D)



E)



t (sec)	0	2	4	6
$a(t)$ (ft/sec^2)	5	2	8	3

91. The data for the acceleration $a(t)$ of a car from 0 to 6 seconds are given in the table above. If the velocity at $t = 0$ is 11 feet per second, the approximate value of the velocity at $t = 6$, computed using a left-hand Riemann sum with three subintervals of equal length, is

(A) 26 ft/sec (B) 30 ft/sec (C) 37 ft/sec (D) 39 ft/sec (E) 41 ft/sec

---

92. Let f be the function given by $f(x) = x^2 - 2x + 3$. The tangent line to the graph of f at $x = 2$ is used to approximate values of $f(x)$. Which of the following is the greatest value of x for which the error resulting from this tangent line approximation is less than 0.5?

(A) 2.4 (B) 2.5 (C) 2.6 (D) 2.7 (E) 2.8

1998 BC

1. C
2. A
3. D
4. A
5. A
6. E
7. E
8. B
9. D
10. E
11. A
12. E
13. B
14. E
15. B
16. C
17. D
18. B
19. D
20. E
21. C
22. A
23. E
24. C
25. C
26. E
27. D
28. C
76. D
77. E
78. B
79. A
80. B
81. B
82. B
83. C
84. B
85. C
86. C
87. D
88. C
89. A
90. A
91. E
92. D

2004 Form B

CALCULUS BC
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.

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1. A particle moving along a curve in the plane has position $(x(t), y(t))$ at time t, where

$$\frac{dx}{dt} = \sqrt{t^4 + 9} \text{ and } \frac{dy}{dt} = 2e^t + 5e^{-t}$$

for all real values of t. At time $t = 0$, the particle is at the point (4, 1).

(a) Find the speed of the particle and its acceleration vector at time $t = 0$.

(b) Find an equation of the line tangent to the path of the particle at time $t = 0$.

(c) Find the total distance traveled by the particle over the time interval $0 \le t \le 3$.

(d) Find the x-coordinate of the position of the particle at time $t = 3$.

2. Let f be a function having derivatives of all orders for all real numbers. The third-degree Taylor polynomial for f about $x = 2$ is given by

$$T(x) = 7 - 9(x-2)^2 - 3(x-2)^3.$$

(a) Find $f(2)$ and $f''(2)$.

(b) Is there enough information given to determine whether f has a critical point at $x = 2$?

If not, explain why not.

If so, determine whether $f(2)$ is a relative maximum, a relative minimum, or neither, and justify your answer.

(c) Use $T(x)$ to find an approximation for $f(0)$. Is there enough information given to determine whether f has a critical point at $x = 0$?

If not, explain why not.

If so, determine whether $f(0)$ is a relative maximum, a relative minimum, or neither, and justify your answer.

(d) The fourth derivative of f satisfies the inequality $\left| f^{(4)}(x) \right| \le 6$ for all x in the closed interval $[0, 2]$. Use the Lagrange error bound on the approximation to $f(0)$ found in part (c) to explain why $f(0)$ is negative.

---

t (minutes)	0	5	10	15	20	25	30	35	40
$v(t)$ (miles per minute)	7.0	9.2	9.5	7.0	4.5	2.4	2.4	4.3	7.3

3. A test plane flies in a straight line with positive velocity $v(t)$, in miles per minute at time t minutes, where v is a differentiable function of t. Selected values of $v(t)$ for $0 \le t \le 40$ are shown in the table above.

(a) Use a midpoint Riemann sum with four subintervals of equal length and values from the table to approximate $\int_0^{40} v(t)\,dt$. Show the computations that lead to your answer. Using correct units, explain the meaning of $\int_0^{40} v(t)\,dt$ in terms of the plane's flight.

(b) Based on the values in the table, what is the smallest number of instances at which the acceleration of the plane could equal zero on the open interval $0 < t < 40$? Justify your answer.

(c) The function f, defined by $f(t) = 6 + \cos\left(\frac{t}{10}\right) + 3\sin\left(\frac{7t}{40}\right)$, is used to model the velocity of the plane, in miles per minute, for $0 \le t \le 40$. According to this model, what is the acceleration of the plane at $t = 23$? Indicate units of measure.

(d) According to the model f, given in part (c), what is the average velocity of the plane, in miles per minute, over the time interval $0 \le t \le 40$?

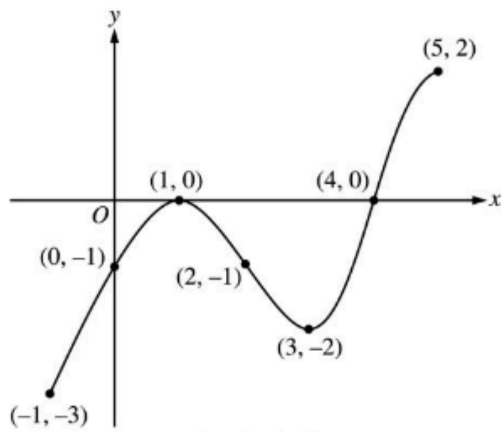
CALCULUS BC

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.



Graph of f'

4. The figure above shows the graph of f', the derivative of the function f, on the closed interval $-1 \le x \le 5$. The graph of f' has horizontal tangent lines at $x = 1$ and $x = 3$. The function f is twice differentiable with $f(2) = 6$.

   (a) Find the x-coordinate of each of the points of inflection of the graph of f. Give a reason for your answer.

   (b) At what value of x does f attain its absolute minimum value on the closed interval $-1 \le x \le 5$? At what value of x does f attain its absolute maximum value on the closed interval $-1 \le x \le 5$? Show the analysis that leads to your answers.

   (c) Let g be the function defined by $g(x) = xf(x)$. Find an equation for the line tangent to the graph of g at $x = 2$.

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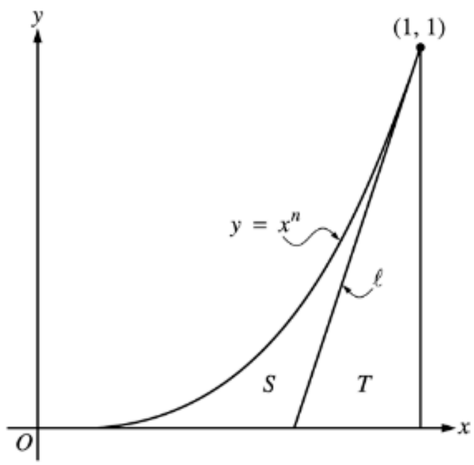
5. Let g be the function given by $g(x) = \dfrac{1}{\sqrt{x}}$.

   (a) Find the average value of g on the closed interval $[1, 4]$.

   (b) Let S be the solid generated when the region bounded by the graph of $y = g(x)$, the vertical lines $x = 1$ and $x = 4$, and the x-axis is revolved about the x-axis. Find the volume of S.

   (c) For the solid S, given in part (b), find the average value of the areas of the cross sections perpendicular to the x-axis.

   (d) The average value of a function f on the unbounded interval $[a, \infty)$ is defined to be $\lim_{b\to\infty}\left[\dfrac{\int_a^b f(x)\,dx}{b-a}\right]$. Show that the improper integral $\int_4^\infty g(x)\,dx$ is divergent, but the average value of g on the interval $[4, \infty)$ is finite.



6. Let ℓ be the line tangent to the graph of $y = x^n$ at the point $(1, 1)$, where $n > 1$, as shown above.

(a) Find $\int_0^1 x^n \, dx$ in terms of n.

(b) Let T be the triangular region bounded by ℓ, the x-axis, and the line $x = 1$. Show that the area of T is $\frac{1}{2n}$.

(c) Let S be the region bounded by the graph of $y = x^n$, the line ℓ, and the x-axis. Express the area of S in terms of n and determine the value of n that maximizes the area of S.

END OF EXAMINATION

Question 1

A particle moving along a curve in the plane has position $(x(t), y(t))$ at time t, where

$$\frac{dx}{dt} = \sqrt{t^4 + 9} \text{ and } \frac{dy}{dt} = 2e^t + 5e^{-t}$$

for all real values of t. At time $t = 0$, the particle is at the point $(4, 1)$.

(a) Find the speed of the particle and its acceleration vector at time $t = 0$.

(b) Find an equation of the line tangent to the path of the particle at time $t = 0$.

(c) Find the total distance traveled by the particle over the time interval $0 \le t \le 3$.

(d) Find the x-coordinate of the position of the particle at time $t = 3$.

(a) At time $t = 0$:

Speed $= \sqrt{x'(0)^2 + y'(0)^2} = \sqrt{3^2 + 7^2} = \sqrt{58}$

Acceleration vector $= \langle x''(0), y''(0) \rangle = \langle 0, -3 \rangle$

$2 : \begin{cases} 1 : \text{speed} \\ 1 : \text{acceleration vector} \end{cases}$

(b) $\frac{dy}{dx} = \frac{y'(0)}{x'(0)} = \frac{7}{3}$

Tangent line is $y = \frac{7}{3}(x - 4) + 1$

$2 : \begin{cases} 1 : \text{slope} \\ 1 : \text{tangent line} \end{cases}$

(c) Distance $= \int_0^3 \sqrt{\left(\sqrt{t^4 + 9}\right)^2 + \left(2e^t + 5e^{-t}\right)^2}\, dt$

$= 45.226$ or 45.227

$3 : \begin{cases} 2 : \text{distance integral} \\ \quad \langle -1 \rangle \text{ each integrand error} \\ \quad \langle -1 \rangle \text{ error in limits} \\ 1 : \text{answer} \end{cases}$

(d) $x(3) = 4 + \int_0^3 \sqrt{t^4 + 9}\, dt$

$= 17.930$ or 17.931

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

Question 2

Let f be a function having derivatives of all orders for all real numbers. The third-degree Taylor polynomial for f about $x = 2$ is given by $T(x) = 7 - 9(x-2)^2 - 3(x-2)^3$.

(a) Find $f(2)$ and $f''(2)$.

(b) Is there enough information given to determine whether f has a critical point at $x = 2$? If not, explain why not. If so, determine whether $f(2)$ is a relative maximum, a relative minimum, or neither, and justify your answer.

(c) Use $T(x)$ to find an approximation for $f(0)$. Is there enough information given to determine whether f has a critical point at $x = 0$? If not, explain why not. If so, determine whether $f(0)$ is a relative maximum, a relative minimum, or neither, and justify your answer.

(d) The fourth derivative of f satisfies the inequality $\left|f^{(4)}(x)\right| \le 6$ for all x in the closed interval $[0, 2]$. Use the Lagrange error bound on the approximation to $f(0)$ found in part (c) to explain why $f(0)$ is negative.

Solution	Scoring
(a) $f(2) = T(2) = 7$ $\dfrac{f''(2)}{2!} = -9$ so $f''(2) = -18$	$2: \begin{cases} 1: f(2) = 7 \\ 1: f''(2) = -18 \end{cases}$
(b) Yes, since $f'(2) = T'(2) = 0$, f does have a critical point at $x = 2$. Since $f''(2) = -18 < 0$, $f(2)$ is a relative maximum value.	$2: \begin{cases} 1: \text{states } f'(2) = 0 \\ 1: \text{declares } f(2) \text{ as a relative maximum because } f''(2) < 0 \end{cases}$
(c) $f(0) \approx T(0) = -5$ It is not possible to determine if f has a critical point at $x = 0$ because $T(x)$ gives exact information only at $x = 2$.	$3: \begin{cases} 1: f(0) \approx T(0) = -5 \\ 1: \text{declares that it is not possible to determine} \\ 1: \text{reason} \end{cases}$
(d) Lagrange error bound $= \dfrac{6}{4!}\lvert 0-2\rvert^4 = 4$ $f(0) \le T(0) + 4 = -1$ Therefore, $f(0)$ is negative.	$2: \begin{cases} 1: \text{value of Lagrange error bound} \\ 1: \text{explanation} \end{cases}$

Question 3

A test plane flies in a straight line with positive velocity $v(t)$, in miles per minute at time t minutes, where v is a differentiable function of t. Selected values of $v(t)$ for $0 \le t \le 40$ are shown in the table above.

t (min)	0	5	10	15	20	25	30	35	40
$v(t)$ (mpm)	7.0	9.2	9.5	7.0	4.5	2.4	2.4	4.3	7.3

(a) Use a midpoint Riemann sum with four subintervals of equal length and values from the table to approximate $\int_0^{40} v(t)\,dt$. Show the computations that lead to your answer. Using correct units, explain the meaning of $\int_0^{40} v(t)\,dt$ in terms of the plane's flight.

(b) Based on the values in the table, what is the smallest number of instances at which the acceleration of the plane could equal zero on the open interval $0 < t < 40$? Justify your answer.

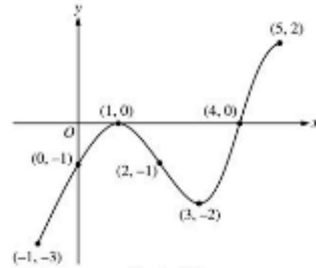
(c) The function f, defined by $f(t) = 6 + \cos\left(\frac{t}{10}\right) + 3\sin\left(\frac{7t}{40}\right)$, is used to model the velocity of the plane, in miles per minute, for $0 \le t \le 40$. According to this model, what is the acceleration of the plane at $t = 23$? Indicates units of measure.

(d) According to the model f, given in part (c), what is the average velocity of the plane, in miles per minute, over the time interval $0 \le t \le 40$?

Solution	Scoring
(a) Midpoint Riemann sum is $10\cdot[v(5)+v(15)+v(25)+v(35)]$ $= 10\cdot[9.2+7.0+2.4+4.3] = 229$. The integral gives the total distance in miles that the plane flies during the 40 minutes.	3 : { 1 : $v(5)+v(15)+v(25)+v(35)$; 1 : answer; 1 : meaning with units }
(b) By the Mean Value Theorem, $v'(t) = 0$ somewhere in the interval $(0, 15)$ and somewhere in the interval $(25, 30)$. Therefore the acceleration will equal 0 for at least two values of t.	2 : { 1 : two instances; 1 : justification }
(c) $f'(23) = -0.407$ or -0.408 miles per minute^2	1 : answer with units
(d) Average velocity $= \frac{1}{40}\int_0^{40} f(t)\,dt$ $= 5.916$ miles per minute	3 : { 1 : limits; 1 : integrand; 1 : answer }

Question 4

The figure above shows the graph of f', the derivative of the function f, on the closed interval $-1 \le x \le 5$. The graph of f' has horizontal tangent lines at $x = 1$ and $x = 3$. The function f is twice differentiable with $f(2) = 6$.



Graph of f'

(a) Find the x-coordinate of each of the points of inflection of the graph of f. Give a reason for your answer.

(b) At what value of x does f attain its absolute minimum value on the closed interval $-1 \le x \le 5$? At what value of x does f attain its absolute maximum value on the closed interval $-1 \le x \le 5$? Show the analysis that leads to your answers.

(c) Let g be the function defined by $g(x) = xf(x)$. Find an equation for the line tangent to the graph of g at $x = 2$.

(a) $x = 1$ and $x = 3$ because the graph of f' changes from increasing to decreasing at $x = 1$, and changes from decreasing to increasing at $x = 3$.

$$2: \begin{cases} 1: x = 1, x = 3 \\ 1: \text{reason} \end{cases}$$

(b) The function f decreases from $x = -1$ to $x = 4$, then increases from $x = 4$ to $x = 5$.
Therefore, the absolute minimum value for f is at $x = 4$.
The absolute maximum value must occur at $x = -1$ or at $x = 5$.

$$f(5) - f(-1) = \int_{-1}^{5} f'(t)\,dt < 0$$

Since $f(5) < f(-1)$, the absolute maximum value occurs at $x = -1$.

$$4: \begin{cases} 1: \text{indicates } f \text{ decreases then increases} \\ 1: \text{eliminates } x = 5 \text{ for maximum} \\ 1: \text{absolute minimum at } x = 4 \\ 1: \text{absolute maximum at } x = -1 \end{cases}$$

(c) $g'(x) = f(x) + xf'(x)$
$g'(2) = f(2) + 2f'(2) = 6 + 2(-1) = 4$
$g(2) = 2f(2) = 12$

Tangent line is $y = 4(x - 2) + 12$

$$3: \begin{cases} 2: g'(x) \\ 1: \text{tangent line} \end{cases}$$

Question 5

Let g be the function given by $g(x) = \dfrac{1}{\sqrt{x}}$.

(a) Find the average value of g on the closed interval $[1, 4]$.

(b) Let S be the solid generated when the region bounded by the graph of $y = g(x)$, the vertical lines $x = 1$ and $x = 4$, and the x-axis is revolved about the x-axis. Find the volume of S.

(c) For the solid S, given in part (b), find the average value of the areas of the cross sections perpendicular to the x-axis.

(d) The average value of a function f on the unbounded interval $[a, \infty)$ is defined to be $\displaystyle\lim_{b\to\infty}\left[\frac{\int_a^b f(x)\,dx}{b-a}\right]$. Show that the improper integral $\displaystyle\int_4^\infty g(x)\,dx$ is divergent, but the average value of g on the interval $[4, \infty)$ is finite.

(a) $$\frac{1}{3}\int_1^4 \frac{1}{\sqrt{x}}\,dx = \frac{1}{3}\cdot 2\sqrt{x}\Big|_1^4 = \frac{4}{3} - \frac{2}{3} = \frac{2}{3}$$

$2: \begin{cases} 1: \text{integral} \\ 1: \text{antidifferentiation and evaluation} \end{cases}$

(b) $$\text{Volume} = \pi\int_1^4 \frac{1}{x}\,dx = \pi \ln x\Big|_1^4 = \pi\ln 4$$

$2: \begin{cases} 1: \text{integral} \\ 1: \text{antidifferentiation and evaluation} \end{cases}$

(c) The cross section at x has area $\pi\left(\dfrac{1}{\sqrt{x}}\right)^2 = \dfrac{\pi}{x}$

$$\text{Average value} = \frac{1}{3}\int_1^4 \frac{\pi}{x}\,dx = \frac{1}{3}\pi\ln 4$$

$1: \text{answer}$

(d) $$\int_4^\infty g(x)\,dx = \lim_{b\to\infty}\int_4^b \frac{1}{\sqrt{x}}\,dx = \lim_{b\to\infty}\left(2\sqrt{b} - 4\right) = \infty$$

This limit is not finite, so the integral is divergent.

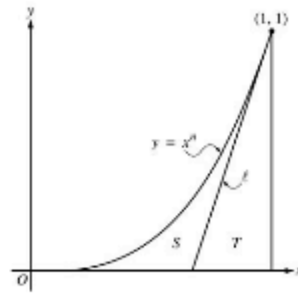
$$\frac{\int_4^b g(x)\,dx}{b-4} = \frac{1}{b-4}\int_4^b \frac{1}{\sqrt{x}}\,dx = \frac{2\sqrt{b}-4}{b-4}$$

$$\lim_{b\to\infty}\frac{2\sqrt{b}-4}{b-4} = 0$$

$4: \begin{cases} 1: \int_4^b g(x)\,dx = 2\sqrt{b} - 4 \\ 1: \text{indicates integral diverges} \\ 1: \dfrac{1}{b-4}\int_4^b g(x)\,dx = \dfrac{2\sqrt{b}-4}{b-4} \\ 1: \text{finite limit as } b \to \infty \end{cases}$

Question 6

Let ℓ be the line tangent to the graph of $y = x^n$ at the point $(1, 1)$, where $n > 1$, as shown above.



(a) Find $\int_0^1 x^n\,dx$ in terms of n.

(b) Let T be the triangular region bounded by ℓ, the x-axis, and the line $x = 1$. Show that the area of T is $\frac{1}{2n}$.

(c) Let S be the region bounded by the graph of $y = x^n$, the line ℓ, and the x-axis. Express the area of S in terms of n and determine the value of n that maximizes the area of S.

(a) $\int_0^1 x^n\,dx = \left.\frac{x^{n+1}}{n+1}\right|_0^1 = \frac{1}{n+1}$

$2: \begin{cases} 1: \text{antiderivative of } x^n \\ 1: \text{answer} \end{cases}$

(b) Let b be the length of the base of triangle T.

$\frac{1}{b}$ is the slope of line ℓ, which is n

$\text{Area}(T) = \frac{1}{2}b(1) = \frac{1}{2n}$

$3: \begin{cases} 1: \text{slope of line } \ell \text{ is } n \\ 1: \text{base of } T \text{ is } \frac{1}{n} \\ 1: \text{shows area is } \frac{1}{2n} \end{cases}$

(c) $\text{Area}(S) = \int_0^1 x^n\,dx - \text{Area}(T)$

$= \frac{1}{n+1} - \frac{1}{2n}$

$\frac{d}{dn}\text{Area}(S) = -\frac{1}{(n+1)^2} + \frac{1}{2n^2} = 0$

$2n^2 = (n+1)^2$

$\sqrt{2}\,n = (n+1)$

$n = \frac{1}{\sqrt{2}-1} = 1 + \sqrt{2}$

$4: \begin{cases} 1: \text{area of } S \text{ in terms of } n \\ 1: \text{derivative} \\ 1: \text{sets derivative equal to } 0 \\ 1: \text{solves for } n \end{cases}$

2004

CALCULUS BC
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.

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1. Traffic flow is defined as the rate at which cars pass through an intersection, measured in cars per minute. The traffic flow at a particular intersection is modeled by the function F defined by

$$F(t) = 82 + 4\sin\left(\frac{t}{2}\right) \text{ for } 0 \le t \le 30,$$

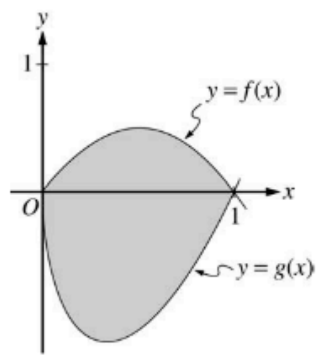
where $F(t)$ is measured in cars per minute and t is measured in minutes.

(a) To the nearest whole number, how many cars pass through the intersection over the 30-minute period?

(b) Is the traffic flow increasing or decreasing at $t = 7$? Give a reason for your answer.

(c) What is the average value of the traffic flow over the time interval $10 \le t \le 15$? Indicate units of measure.

(d) What is the average rate of change of the traffic flow over the time interval $10 \le t \le 15$? Indicate units of measure.



2. Let f and g be the functions given by $f(x) = 2x(1-x)$ and $g(x) = 3(x-1)\sqrt{x}$ for $0 \le x \le 1$. The graphs of f and g are shown in the figure above.

(a) Find the area of the shaded region enclosed by the graphs of f and g.

(b) Find the volume of the solid generated when the shaded region enclosed by the graphs of f and g is revolved about the horizontal line $y = 2$.

(c) Let h be the function given by $h(x) = kx(1-x)$ for $0 \le x \le 1$. For each $k > 0$, the region (not shown) enclosed by the graphs of h and g is the base of a solid with square cross sections perpendicular to the x-axis. There is a value of k for which the volume of this solid is equal to 15. Write, but do not solve, an equation involving an integral expression that could be used to find the value of k.

---

3. An object moving along a curve in the xy-plane has position $(x(t), y(t))$ at time $t \geq 0$ with $\frac{dx}{dt} = 3 + \cos(t^2)$. The derivative $\frac{dy}{dt}$ is not explicitly given. At time $t = 2$, the object is at position $(1, 8)$.

(a) Find the x-coordinate of the position of the object at time $t = 4$.

(b) At time $t = 2$, the value of $\frac{dy}{dt}$ is -7. Write an equation for the line tangent to the curve at the point $(x(2), y(2))$.

(c) Find the speed of the object at time $t = 2$.

(d) For $t \geq 3$, the line tangent to the curve at $(x(t), y(t))$ has a slope of $2t + 1$. Find the acceleration vector of the object at time $t = 4$.

No calculator is allowed for these problems.

---

4. Consider the curve given by $x^2 + 4y^2 = 7 + 3xy$.

(a) Show that $\frac{dy}{dx} = \frac{3y - 2x}{8y - 3x}$.

(b) Show that there is a point P with x-coordinate 3 at which the line tangent to the curve at P is horizontal. Find the y-coordinate of P.

(c) Find the value of $\frac{d^2y}{dx^2}$ at the point P found in part (b). Does the curve have a local maximum, a local minimum, or neither at the point P? Justify your answer.

---

5. A population is modeled by a function P that satisfies the logistic differential equation

$$\frac{dP}{dt} = \frac{P}{5}\left(1 - \frac{P}{12}\right).$$

(a) If $P(0) = 3$, what is $\lim_{t \to \infty} P(t)$?

If $P(0) = 20$, what is $\lim_{t \to \infty} P(t)$?

(b) If $P(0) = 3$, for what value of P is the population growing the fastest?

(c) A different population is modeled by a function Y that satisfies the separable differential equation

$$\frac{dY}{dt} = \frac{Y}{5}\left(1 - \frac{t}{12}\right).$$

Find $Y(t)$ if $Y(0) = 3$.

(d) For the function Y found in part (c), what is $\lim_{t \to \infty} Y(t)$?

---

6. Let f be the function given by $f(x) = \sin\left(5x + \frac{\pi}{4}\right)$, and let $P(x)$ be the third-degree Taylor polynomial for f about $x = 0$.

(a) Find $P(x)$.

(b) Find the coefficient of x^{22} in the Taylor series for f about $x = 0$.

(c) Use the Lagrange error bound to show that $\left| f\left(\frac{1}{10}\right) - P\left(\frac{1}{10}\right) \right| < \frac{1}{100}$.

(d) Let G be the function given by $G(x) = \int_0^x f(t)\,dt$. Write the third-degree Taylor polynomial for G about $x = 0$.

END OF EXAMINATION

Question 1

Traffic flow is defined as the rate at which cars pass through an intersection, measured in cars per minute. The traffic flow at a particular intersection is modeled by the function F defined by

$$F(t) = 82 + 4\sin\left(\frac{t}{2}\right) \text{ for } 0 \le t \le 30,$$

where $F(t)$ is measured in cars per minute and t is measured in minutes.

(a) To the nearest whole number, how many cars pass through the intersection over the 30-minute period?

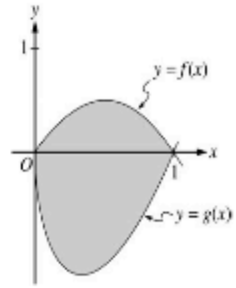
(b) Is the traffic flow increasing or decreasing at $t = 7$? Give a reason for your answer.

(c) What is the average value of the traffic flow over the time interval $10 \le t \le 15$? Indicate units of measure.

(d) What is the average rate of change of the traffic flow over the time interval $10 \le t \le 15$? Indicate units of measure.

(a) $\int_0^{30} F(t)\,dt = 2474$ cars	$3: \begin{cases} 1: \text{limits} \\ 1: \text{integrand} \\ 1: \text{answer} \end{cases}$
(b) $F'(7) = -1.872$ or -1.873 Since $F'(7) < 0$, the traffic flow is decreasing at $t = 7$.	1 : answer with reason
(c) $\frac{1}{5}\int_{10}^{15} F(t)\,dt = 81.899$ cars/min	$3: \begin{cases} 1: \text{limits} \\ 1: \text{integrand} \\ 1: \text{answer} \end{cases}$
(d) $\dfrac{F(15) - F(10)}{15 - 10} = 1.517$ or 1.518 cars/min^2	1 : answer
Units of cars/min in (c) and cars/min^2 in (d)	1 : units in (c) and (d)

Question 2



Let f and g be the functions given by $f(x) = 2x(1 - x)$ and $g(x) = 3(x - 1)\sqrt{x}$ for $0 \le x \le 1$. The graphs of f and g are shown in the figure above.

(a) Find the area of the shaded region enclosed by the graphs of f and g.

(b) Find the volume of the solid generated when the shaded region enclosed by the graphs of f and g is revolved about the horizontal line $y = 2$.

(c) Let h be the function given by $h(x) = kx(1 - x)$ for $0 \le x \le 1$. For each $k > 0$, the region (not shown) enclosed by the graphs of h and g is the base of a solid with square cross sections perpendicular to the x-axis. There is a value of k for which the volume of this solid is equal to 15. Write, but do not solve, an equation involving an integral expression that could be used to find the value of k.

(a) Area $= \int_0^1 (f(x) - g(x))\,dx$

$= \int_0^1 \left(2x(1 - x) - 3(x - 1)\sqrt{x}\right)dx = 1.133$

$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$

(b) Volume $= \pi \int_0^1 \left((2 - g(x))^2 - (2 - f(x))^2\right)dx$

$= \pi \int_0^1 \left(\left(2 - 3(x - 1)\sqrt{x}\right)^2 - \left(2 - 2x(1 - x)\right)^2\right)dx$

$= 16.179$

$4: \begin{cases} 1: \text{limits and constant} \\ 2: \text{integrand} \\ \quad \langle -1 \rangle \text{ each error} \\ \quad \text{Note: } 0/2 \text{ if integral not of form} \\ \qquad c\int_a^b \left(R^2(x) - r^2(x)\right)dx \\ 1: \text{answer} \end{cases}$

(c) Volume $= \int_0^1 (h(x) - g(x))^2\,dx$

$\int_0^1 \left(kx(1 - x) - 3(x - 1)\sqrt{x}\right)^2 dx = 15$

$3: \begin{cases} 2: \text{integrand} \\ 1: \text{answer} \end{cases}$

Question 3

An object moving along a curve in the xy-plane has position $(x(t), y(t))$ at time $t \geq 0$ with $\frac{dx}{dt} = 3 + \cos(t^2)$. The derivative $\frac{dy}{dt}$ is not explicitly given. At time $t = 2$, the object is at position $(1, 8)$.

(a) Find the x-coordinate of the position of the object at time $t = 4$.

(b) At time $t = 2$, the value of $\frac{dy}{dt}$ is -7. Write an equation for the line tangent to the curve at the point $(x(2), y(2))$.

(c) Find the speed of the object at time $t = 2$.

(d) For $t \geq 3$, the line tangent to the curve at $(x(t), y(t))$ has a slope of $2t + 1$. Find the acceleration vector of the object at time $t = 4$.

(a) $x(4) = x(2) + \int_2^4 \left(3 + \cos\left(t^2\right)\right) dt$ $= 1 + \int_2^4 \left(3 + \cos\left(t^2\right)\right) dt = 7.132 \text{ or } 7.133$	$3: \begin{cases} 1: \int_2^4 \left(3 + \cos\left(t^2\right)\right) dt \\ 1: \text{handles initial condition} \\ 1: \text{answer} \end{cases}$
(b) $\left.\frac{dy}{dx}\right\|_{t=2} = \left.\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right\|_{t=2} = \frac{-7}{3 + \cos 4} = -2.983$ $y - 8 = -2.983(x - 1)$	$2: \begin{cases} 1: \text{finds } \left.\frac{dy}{dx}\right\|_{t=2} \\ 1: \text{equation} \end{cases}$
(c) The speed of the object at time $t = 2$ is $\sqrt{(x'(2))^2 + (y'(2))^2} = 7.382 \text{ or } 7.383.$	$1: \text{answer}$
(d) $x''(4) = 2.303$ $y'(t) = \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = (2t + 1)\left(3 + \cos\left(t^2\right)\right)$ $y''(4) = 24.813 \text{ or } 24.814$ The acceleration vector at $t = 4$ is $\langle 2.303, 24.813 \rangle$ or $\langle 2.303, 24.814 \rangle$.	$3: \begin{cases} 1: x''(4) \\ 1: \frac{dy}{dt} \\ 1: \text{answer} \end{cases}$

Question 4

Consider the curve given by $x^2 + 4y^2 = 7 + 3xy$.

(a) Show that $\frac{dy}{dx} = \frac{3y - 2x}{8y - 3x}$.

(b) Show that there is a point P with x-coordinate 3 at which the line tangent to the curve at P is horizontal. Find the y-coordinate of P.

(c) Find the value of $\frac{d^2y}{dx^2}$ at the point P found in part (b). Does the curve have a local maximum, a local minimum, or neither at the point P? Justify your answer.

Solution	Scoring
(a) $2x + 8yy' = 3y + 3xy'$ $(8y - 3x)y' = 3y - 2x$ $y' = \frac{3y - 2x}{8y - 3x}$	$2: \begin{cases} 1: \text{implicit differentiation} \\ 1: \text{solves for } y' \end{cases}$
(b) $\frac{3y - 2x}{8y - 3x} = 0;\ 3y - 2x = 0$ When $x = 3,\ 3y = 6$ $y = 2$ $3^2 + 4 \cdot 2^2 = 25$ and $7 + 3 \cdot 3 \cdot 2 = 25$ Therefore, $P = (3, 2)$ is on the curve and the slope is 0 at this point.	$3: \begin{cases} 1: \frac{dy}{dx} = 0 \\ 1: \text{shows slope is 0 at } (3, 2) \\ 1: \text{shows } (3, 2) \text{ lies on curve} \end{cases}$
(c) $\frac{d^2y}{dx^2} = \frac{(8y - 3x)(3y' - 2) - (3y - 2x)(8y' - 3)}{(8y - 3x)^2}$ At $P = (3, 2)$, $\frac{d^2y}{dx^2} = \frac{(16 - 9)(-2)}{(16 - 9)^2} = -\frac{2}{7}$. Since $y' = 0$ and $y'' < 0$ at P, the curve has a local maximum at P.	$4: \begin{cases} 2: \frac{d^2y}{dx^2} \\ 1: \text{value of } \frac{d^2y}{dx^2} \text{ at } (3, 2) \\ 1: \text{conclusion with justification} \end{cases}$

Question 5

A population is modeled by a function P that satisfies the logistic differential equation

$$\frac{dP}{dt} = \frac{P}{5}\left(1 - \frac{P}{12}\right).$$

(a) If $P(0) = 3$, what is $\lim_{t \to \infty} P(t)$?

If $P(0) = 20$, what is $\lim_{t \to \infty} P(t)$?

(b) If $P(0) = 3$, for what value of P is the population growing the fastest?

(c) A different population is modeled by a function Y that satisfies the separable differential equation

$$\frac{dY}{dt} = \frac{Y}{5}\left(1 - \frac{t}{12}\right).$$

Find $Y(t)$ if $Y(0) = 3$.

(d) For the function Y found in part (c), what is $\lim_{t \to \infty} Y(t)$?

(a) For this logistic differential equation, the carrying capacity is 12. If $P(0) = 3$, $\lim_{t \to \infty} P(t) = 12$. If $P(0) = 20$, $\lim_{t \to \infty} P(t) = 12$.	$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{answer} \end{cases}$
(b) The population is growing the fastest when P is half the carrying capacity. Therefore, P is growing the fastest when $P = 6$.	1 : answer
(c) $\frac{1}{Y}dY = \frac{1}{5}\left(1 - \frac{t}{12}\right)dt = \left(\frac{1}{5} - \frac{t}{60}\right)dt$ $\ln\lvert Y\rvert = \frac{t}{5} - \frac{t^2}{120} + C$ $Y(t) = Ke^{\frac{t}{5} - \frac{t^2}{120}}$ $K = 3$ $Y(t) = 3e^{\frac{t}{5} - \frac{t^2}{120}}$	$5 : \begin{cases} 1 : \text{separates variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } Y \\ \quad 0/1 \text{ if } Y \text{ is not exponential} \end{cases}$ Note: max 2/5 [1-1-0-0-0] if no constant of integration Note: 0/5 if no separation of variables
(d) $\lim_{t \to \infty} Y(t) = 0$	1 : answer 0/1 if Y is not exponential

Question 6

Let f be the function given by $f(x) = \sin\left(5x + \frac{\pi}{4}\right)$, and let $P(x)$ be the third-degree Taylor polynomial for f about $x = 0$.

(a) Find $P(x)$.

(b) Find the coefficient of x^{22} in the Taylor series for f about $x = 0$.

(c) Use the Lagrange error bound to show that $\left|f\left(\frac{1}{10}\right) - P\left(\frac{1}{10}\right)\right| < \frac{1}{100}$.

(d) Let G be the function given by $G(x) = \int_0^x f(t)\,dt$. Write the third-degree Taylor polynomial for G about $x = 0$.

(a) $f(0) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ $f'(0) = 5\cos\left(\frac{\pi}{4}\right) = \frac{5\sqrt{2}}{2}$ $f''(0) = -25\sin\left(\frac{\pi}{4}\right) = -\frac{25\sqrt{2}}{2}$ $f'''(0) = -125\cos\left(\frac{\pi}{4}\right) = -\frac{125\sqrt{2}}{2}$ $P(x) = \frac{\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}x - \frac{25\sqrt{2}}{2(2!)}x^2 - \frac{125\sqrt{2}}{2(3!)}x^3$	4 : $P(x)$ $\langle -1 \rangle$ each error or missing term deduct only once for $\sin\left(\frac{\pi}{4}\right)$ evaluation error deduct only once for $\cos\left(\frac{\pi}{4}\right)$ evaluation error $\langle -1 \rangle$ max for all extra terms, $+\cdots$, misuse of equality
(b) $\frac{-5^{22}\sqrt{2}}{2(22!)}$	2 : $\begin{cases} 1 : \text{magnitude} \\ 1 : \text{sign} \end{cases}$
(c) $\left|f\left(\frac{1}{10}\right) - P\left(\frac{1}{10}\right)\right| \le \max\limits_{0 \le c \le \frac{1}{10}} \left|f^{(4)}(c)\right| \left(\frac{1}{4!}\right)\left(\frac{1}{10}\right)^4$ $\le \frac{625}{4!}\left(\frac{1}{10}\right)^4 = \frac{1}{384} < \frac{1}{100}$	1 : error bound in an appropriate inequality
(d) The third-degree Taylor polynomial for G about $x = 0$ is $\int_0^x \left(\frac{\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}t - \frac{25\sqrt{2}}{4}t^2\right) dt$ $= \frac{\sqrt{2}}{2}x + \frac{5\sqrt{2}}{4}x^2 - \frac{25\sqrt{2}}{12}x^3$	2 : third-degree Taylor polynomial for G about $x = 0$ $\langle -1 \rangle$ each incorrect or missing term $\langle -1 \rangle$ max for all extra terms, $+\cdots$, misuse of equality

2005 AP® CALCULUS BC FREE-RESPONSE QUESTIONS (Form B)

CALCULUS BC
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.

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1. An object moving along a curve in the xy-plane has position $(x(t), y(t))$ at time $t \geq 0$ with

$$\frac{dx}{dt} = 12t - 3t^2 \text{ and } \frac{dy}{dt} = \ln\left(1 + (t-4)^4\right).$$

At time $t = 0$, the object is at position $(-13, 5)$. At time $t = 2$, the object is at point P with x-coordinate 3.

(a) Find the acceleration vector at time $t = 2$ and the speed at time $t = 2$.

(b) Find the y-coordinate of P.

(c) Write an equation for the line tangent to the curve at P.

(d) For what value of t, if any, is the object at rest? Explain your reasoning.

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2. A water tank at Camp Newton holds 1200 gallons of water at time $t = 0$. During the time interval $0 \leq t \leq 18$ hours, water is pumped into the tank at the rate

$$W(t) = 95\sqrt{t}\sin^2\left(\frac{t}{6}\right) \text{ gallons per hour.}$$

During the same time interval, water is removed from the tank at the rate

$$R(t) = 275\sin^2\left(\frac{t}{3}\right) \text{ gallons per hour.}$$

(a) Is the amount of water in the tank increasing at time $t = 15$? Why or why not?

(b) To the nearest whole number, how many gallons of water are in the tank at time $t = 18$?

(c) At what time t, for $0 \leq t \leq 18$, is the amount of water in the tank at an absolute minimum? Show the work that leads to your conclusion.

(d) For $t > 18$, no water is pumped into the tank, but water continues to be removed at the rate $R(t)$ until the tank becomes empty. Let k be the time at which the tank becomes empty. Write, but do not solve, an equation involving an integral expression that can be used to find the value of k.

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3. The Taylor series about $x = 0$ for a certain function f converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x = 0$ is given by

$$f^{(n)}(0) = \frac{(-1)^{n+1}(n+1)!}{5^n (n-1)^2} \text{ for } n \geq 2.$$

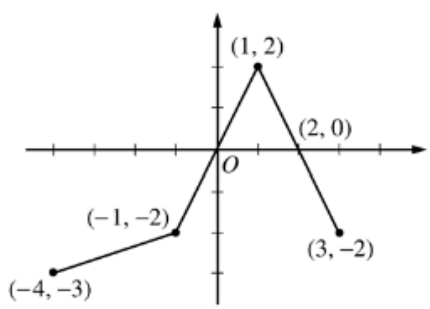
The graph of f has a horizontal tangent line at $x = 0$, and $f(0) = 6$.

(a) Determine whether f has a relative maximum, a relative minimum, or neither at $x = 0$. Justify your answer.

(b) Write the third-degree Taylor polynomial for f about $x = 0$.

(c) Find the radius of convergence of the Taylor series for f about $x = 0$. Show the work that leads to your answer.

No calculator is allowed for these problems.



Graph of f

4. The graph of the function f above consists of three line segments.

(a) Let g be the function given by $g(x) = \int_{-4}^{x} f(t)\,dt$. For each of $g(-1)$, $g'(-1)$, and $g''(-1)$, find the value or state that it does not exist.

(b) For the function g defined in part (a), find the x-coordinate of each point of inflection of the graph of g on the open interval $-4 < x < 3$. Explain your reasoning.

(c) Let h be the function given by $h(x) = \int_{x}^{3} f(t)\,dt$. Find all values of x in the closed interval $-4 \leq x \leq 3$ for which $h(x) = 0$.

(d) For the function h defined in part (c), find all intervals on which h is decreasing. Explain your reasoning.

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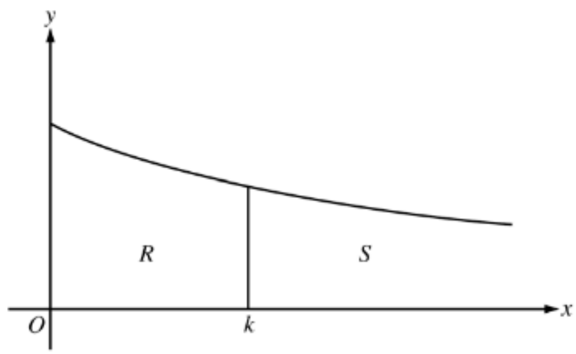
5. Consider the curve given by $y^2 = 2 + xy$.

(a) Show that $\frac{dy}{dx} = \frac{y}{2y - x}$.

(b) Find all points (x, y) on the curve where the line tangent to the curve has slope $\frac{1}{2}$.

(c) Show that there are no points (x, y) on the curve where the line tangent to the curve is horizontal.

(d) Let x and y be functions of time t that are related by the equation $y^2 = 2 + xy$. At time $t = 5$, the value of y is 3 and $\frac{dy}{dt} = 6$. Find the value of $\frac{dx}{dt}$ at time $t = 5$.



6. Consider the graph of the function f given by $f(x) = \frac{1}{x+2}$ for $x \geq 0$, as shown in the figure above. Let R be the region bounded by the graph of f, the x- and y-axes, and the vertical line $x = k$, where $k \geq 0$.

(a) Find the area of R in terms of k.

(b) Find the volume of the solid generated when R is revolved about the x-axis in terms of k.

(c) Let S be the unbounded region in the first quadrant to the right of the vertical line $x = k$ and below the graph of f, as shown in the figure above. Find all values of k such that the volume of the solid generated when S is revolved about the x-axis is equal to the volume of the solid found in part (b).

WRITE ALL WORK IN THE TEST BOOKLET.

END OF EXAM

Question 1

An object moving along a curve in the xy-plane has position $(x(t), y(t))$ at time $t \geq 0$ with

$$\frac{dx}{dt} = 12t - 3t^2 \text{ and } \frac{dy}{dt} = \ln\left(1 + (t-4)^4\right).$$

At time $t = 0$, the object is at position $(-13, 5)$. At time $t = 2$, the object is at point P with x-coordinate 3.

(a) Find the acceleration vector at time $t = 2$ and the speed at time $t = 2$.

(b) Find the y-coordinate of P.

(c) Write an equation for the line tangent to the curve at P.

(d) For what value of t, if any, is the object at rest? Explain your reasoning.

Solution	Scoring
(a) $x''(2) = 0,\ y''(2) = -\frac{32}{17} = -1.882$ $a(2) = \langle 0, -1.882 \rangle$ Speed $= \sqrt{12^2 + (\ln(17))^2} = 12.329$ or 12.330	$2: \begin{cases} 1: \text{acceleration vector} \\ 1: \text{speed} \end{cases}$
(b) $y(t) = y(0) + \int_0^t \ln\left(1 + (u-4)^4\right)du$ $y(2) = 5 + \int_0^2 \ln\left(1 + (u-4)^4\right)du = 13.671$	$3: \begin{cases} 1: \int_0^2 \ln\left(1 + (u-4)^4\right)du \\ 1: \text{handles initial condition} \\ 1: \text{answer} \end{cases}$
(c) At $t = 2$, slope $= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\ln(17)}{12} = 0.236$ $y - 13.671 = 0.236(x - 3)$	$2: \begin{cases} 1: \text{slope} \\ 1: \text{equation} \end{cases}$
(d) $x'(t) = 0$ if $t = 0, 4$ $y'(t) = 0$ if $t = 4$ $t = 4$	$2: \begin{cases} 1: \text{reason} \\ 1: \text{answer} \end{cases}$

Question 2

A water tank at Camp Newton holds 1200 gallons of water at time $t = 0$. During the time interval $0 \le t \le 18$ hours, water is pumped into the tank at the rate

$$W(t) = 95\sqrt{t}\sin^2\left(\frac{t}{6}\right) \text{ gallons per hour.}$$

During the same time interval, water is removed from the tank at the rate

$$R(t) = 275\sin^2\left(\frac{t}{3}\right) \text{ gallons per hour.}$$

(a) Is the amount of water in the tank increasing at time $t = 15$? Why or why not?

(b) To the nearest whole number, how many gallons of water are in the tank at time $t = 18$?

(c) At what time t, for $0 \le t \le 18$, is the amount of water in the tank at an absolute minimum? Show the work that leads to your conclusion.

(d) For $t > 18$, no water is pumped into the tank, but water continues to be removed at the rate $R(t)$ until the tank becomes empty. Let k be the time at which the tank becomes empty. Write, but do not solve, an equation involving an integral expression that can be used to find the value of k.

(a) No; the amount of water is not increasing at $t = 15$ since $W(15) - R(15) = -121.09 < 0$.

1 : answer with reason

(b) $1200 + \int_0^{18} (W(t) - R(t))\,dt = 1309.788$

1310 gallons

3 : { 1 : limits; 1 : integrand; 1 : answer }

(c) $W(t) - R(t) = 0$

$t = 0, 6.4948, 12.9748$

t (hours)	gallons of water
0	1200
6.495	525
12.975	1697
18	1310

The values at the endpoints and the critical points show that the absolute minimum occurs when $t = 6.494$ or 6.495.

3 : { 1 : interior critical points; 1 : amount of water is least at $t = 6.494$ or 6.495; 1 : analysis for absolute minimum }

(d) $\int_{18}^{k} R(t)\,dt = 1310$

2 : { 1 : limits; 1 : equation }

Question 3

The Taylor series about $x = 0$ for a certain function f converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x = 0$ is given by

$$f^{(n)}(0) = \frac{(-1)^{n+1}(n+1)!}{5^n(n-1)^2} \text{ for } n \geq 2.$$

The graph of f has a horizontal tangent line at $x = 0$, and $f(0) = 6$.

(a) Determine whether f has a relative maximum, a relative minimum, or neither at $x = 0$. Justify your answer.

(b) Write the third-degree Taylor polynomial for f about $x = 0$.

(c) Find the radius of convergence of the Taylor series for f about $x = 0$. Show the work that leads to your answer.

(a) f has a relative maximum at $x = 0$ because $f'(0) = 0$ and $f''(0) < 0$.

$2: \begin{cases} 1: \text{answer} \\ 1: \text{reason} \end{cases}$

(b) $f(0) = 6,\ f'(0) = 0$

$$f''(0) = -\frac{3!}{5^2 1^2} = -\frac{6}{25},\ f'''(0) = \frac{4!}{5^3 2^2}$$

$$P(x) = 6 - \frac{3!x^2}{5^2 2!} + \frac{4!x^3}{5^3 2^2 3!} = 6 - \frac{3}{25}x^2 + \frac{1}{125}x^3$$

3 : $P(x)$
$\langle -1 \rangle$ each incorrect term
Note: $\langle -1 \rangle$ max for use of extra terms

(c) $$u_n = \frac{f^{(n)}(0)}{n!}x^n = \frac{(-1)^{n+1}(n+1)}{5^n(n-1)^2}x^n$$

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{\dfrac{(-1)^{n+2}(n+2)}{5^{n+1}n^2}x^{n+1}}{\dfrac{(-1)^{n+1}(n+1)}{5^n(n-1)^2}x^n}\right|$$

$$= \left(\frac{n+2}{n+1}\right)\left(\frac{n-1}{n}\right)^2\frac{1}{5}|x|$$

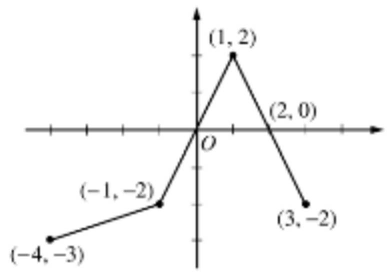
$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \frac{1}{5}|x| < 1 \text{ if } |x| < 5.$$

The radius of convergence is 5.

$4: \begin{cases} 1: \text{general term} \\ 1: \text{sets up ratio} \\ 1: \text{computes limit} \\ 1: \text{applies ratio test to get radius of convergence} \end{cases}$

Question 4

The graph of the function f above consists of three line segments.



Graph of f

(a) Let g be the function given by $g(x) = \int_{-4}^{x} f(t)\,dt$.

For each of $g(-1)$, $g'(-1)$, and $g''(-1)$, find the value or state that it does not exist.

(b) For the function g defined in part (a), find the x-coordinate of each point of inflection of the graph of g on the open interval $-4 < x < 3$. Explain your reasoning.

(c) Let h be the function given by $h(x) = \int_{x}^{3} f(t)\,dt$. Find all values of x in the closed interval $-4 \le x \le 3$ for which $h(x) = 0$.

(d) For the function h defined in part (c), find all intervals on which h is decreasing. Explain your reasoning.

(a) $g(-1) = \int_{-4}^{-1} f(t)\,dt = -\frac{1}{2}(3)(5) = -\frac{15}{2}$ $g'(-1) = f(-1) = -2$ $g''(-1)$ does not exist because f is not differentiable at $x = -1$.	$3: \begin{cases} 1: g(-1) \\ 1: g'(-1) \\ 1: g''(-1) \end{cases}$
(b) $x = 1$ $g' = f$ changes from increasing to decreasing at $x = 1$.	$2: \begin{cases} 1: x = 1 \text{ (only)} \\ 1: \text{reason} \end{cases}$
(c) $x = -1, 1, 3$	2 : correct values $\langle -1 \rangle$ each missing or extra value
(d) h is decreasing on $[0, 2]$ $h' = -f < 0$ when $f > 0$	$2: \begin{cases} 1: \text{interval} \\ 1: \text{reason} \end{cases}$

Question 5

Consider the curve given by $y^2 = 2 + xy$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{y}{2y - x}$.

(b) Find all points (x, y) on the curve where the line tangent to the curve has slope $\dfrac{1}{2}$.

(c) Show that there are no points (x, y) on the curve where the line tangent to the curve is horizontal.

(d) Let x and y be functions of time t that are related by the equation $y^2 = 2 + xy$. At time $t = 5$, the value of y is 3 and $\dfrac{dy}{dt} = 6$. Find the value of $\dfrac{dx}{dt}$ at time $t = 5$.

(a) $2yy' = y + xy'$

$(2y - x)y' = y$

$y' = \dfrac{y}{2y - x}$

$2: \begin{cases} 1: \text{implicit differentiation} \\ 1: \text{solves for } y' \end{cases}$

(b) $\dfrac{y}{2y - x} = \dfrac{1}{2}$

$2y = 2y - x$

$x = 0$

$y = \pm\sqrt{2}$

$(0, \sqrt{2}),\ (0, -\sqrt{2})$

$2: \begin{cases} 1: \dfrac{y}{2y - x} = \dfrac{1}{2} \\ 1: \text{answer} \end{cases}$

(c) $\dfrac{y}{2y - x} = 0$

$y = 0$

The curve has no horizontal tangent since $0^2 \neq 2 + x \cdot 0$ for any x.

$2: \begin{cases} 1: y = 0 \\ 1: \text{explanation} \end{cases}$

(d) When $y = 3$, $3^2 = 2 + 3x$ so $x = \dfrac{7}{3}$.

$\dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dt} = \dfrac{y}{2y - x} \cdot \dfrac{dx}{dt}$

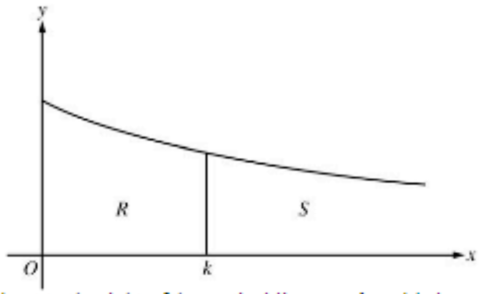
At $t = 5$, $6 = \dfrac{3}{6 - \frac{7}{3}} \cdot \dfrac{dx}{dt} = \dfrac{9}{11} \cdot \dfrac{dx}{dt}$

$\left.\dfrac{dx}{dt}\right|_{t=5} = \dfrac{22}{3}$

$3: \begin{cases} 1: \text{solves for } x \\ 1: \text{chain rule} \\ 1: \text{answer} \end{cases}$

Question 6

Consider the graph of the function f given by $f(x) = \frac{1}{x+2}$ for $x \ge 0$, as shown in the figure above. Let R be the region bounded by the graph of f, the x- and y-axes, and the vertical line $x = k$, where $k \ge 0$.



(a) Find the area of R in terms of k.

(b) Find the volume of the solid generated when R is revolved about the x-axis in terms of k.

(c) Let S be the unbounded region in the first quadrant to the right of the vertical line $x = k$ and below the graph of f, as shown in the figure above. Find all values of k such that the volume of the solid generated when S is revolved about the x-axis is equal to the volume of the solid found in part (b).

(a) Area of $R = \int_0^k \frac{1}{x+2}\,dx = \ln(k+2) - \ln(2)$

$2: \begin{cases} 1: \text{integral} \\ 1: \text{antidifferentiation and evaluation} \end{cases}$

(b) $V_R = \pi \int_0^k \frac{1}{(x+2)^2}\,dx$

$= -\frac{\pi}{x+2}\Big|_0^k = \frac{\pi}{2} - \frac{\pi}{k+2}$

$3: \begin{cases} 1: \text{limits} \\ 1: \text{integrand} \\ 1: \text{antidifferentiation and evaluation} \end{cases}$

(c) $V_S = \pi \int_k^\infty \frac{1}{(x+2)^2}\,dx$

$= \lim_{n\to\infty} -\frac{\pi}{x+2}\Big|_k^n = \frac{\pi}{k+2}$

$V_S = V_R$

$\frac{\pi}{k+2} = \frac{\pi}{2} - \frac{\pi}{k+2}$

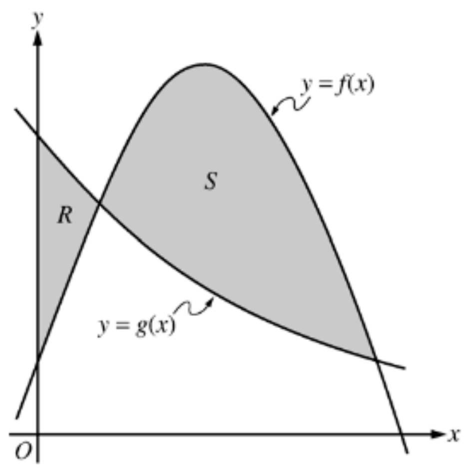
$\frac{2}{k+2} = \frac{1}{2}$

$k = 2$

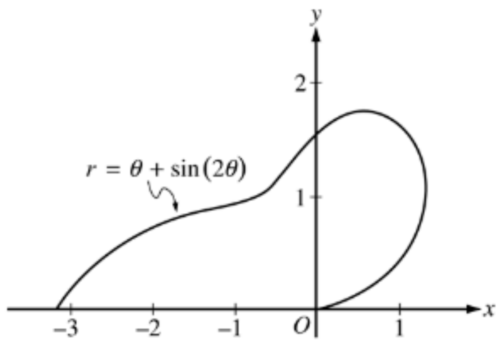
$4: \begin{cases} 1: \text{improper integral} \\ 1: \text{antidifferentiation and evaluation} \\ 1: \text{equation} \\ 1: \text{answer} \end{cases}$

CALCULUS BC
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.



1. Let f and g be the functions given by $f(x) = \frac{1}{4} + \sin(\pi x)$ and $g(x) = 4^{-x}$. Let R be the shaded region in the first quadrant enclosed by the y-axis and the graphs of f and g, and let S be the shaded region in the first quadrant enclosed by the graphs of f and g, as shown in the figure above.
   (a) Find the area of R.
   (b) Find the area of S.
   (c) Find the volume of the solid generated when S is revolved about the horizontal line $y = -1$.



2. The curve above is drawn in the xy-plane and is described by the equation in polar coordinates $r = \theta + \sin(2\theta)$ for $0 \le \theta \le \pi$, where r is measured in meters and θ is measured in radians. The derivative of r with respect to θ is given by $\frac{dr}{d\theta} = 1 + 2\cos(2\theta)$.

   (a) Find the area bounded by the curve and the x-axis.

   (b) Find the angle θ that corresponds to the point on the curve with x-coordinate -2.

   (c) For $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$, $\frac{dr}{d\theta}$ is negative. What does this fact say about r? What does this fact say about the curve?

   (d) Find the value of θ in the interval $0 \le \theta \le \frac{\pi}{2}$ that corresponds to the point on the curve in the first quadrant with greatest distance from the origin. Justify your answer.

Distance x (cm)	0	1	5	6	8
Temperature $T(x)$ (°C)	100	93	70	62	55

3. A metal wire of length 8 centimeters (cm) is heated at one end. The table above gives selected values of the temperature $T(x)$, in degrees Celsius (°C), of the wire x cm from the heated end. The function T is decreasing and twice differentiable.

   (a) Estimate $T'(7)$. Show the work that leads to your answer. Indicate units of measure.

   (b) Write an integral expression in terms of $T(x)$ for the average temperature of the wire. Estimate the average temperature of the wire using a trapezoidal sum with the four subintervals indicated by the data in the table. Indicate units of measure.

   (c) Find $\int_0^8 T'(x)\,dx$, and indicate units of measure. Explain the meaning of $\int_0^8 T'(x)\,dx$ in terms of the temperature of the wire.

   (d) Are the data in the table consistent with the assertion that $T''(x) > 0$ for every x in the interval $0 < x < 8$? Explain your answer.

No calculator is allowed for these problems.

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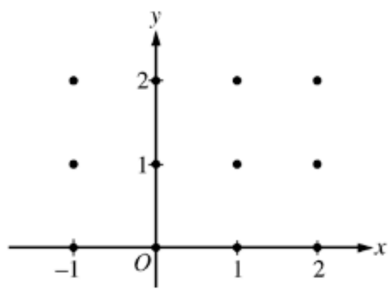
4. Consider the differential equation $\frac{dy}{dx} = 2x - y$.

   (a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated, and sketch the solution curve that passes through the point $(0, 1)$.

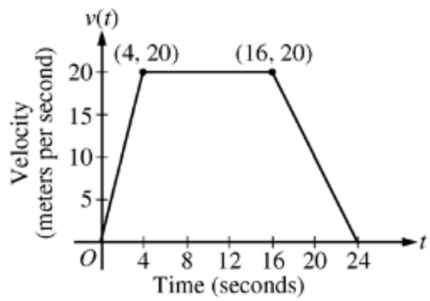
   (Note: Use the axes provided in the pink test booklet.)



   (b) The solution curve that passes through the point $(0, 1)$ has a local minimum at $x = \ln\left(\frac{3}{2}\right)$. What is the y-coordinate of this local minimum?

   (c) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(0) = 1$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f(-0.4)$. Show the work that leads to your answer.

   (d) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine whether the approximation found in part (c) is less than or greater than $f(-0.4)$. Explain your reasoning.



5. A car is traveling on a straight road. For $0 \le t \le 24$ seconds, the car's velocity $v(t)$, in meters per second, is modeled by the piecewise-linear function defined by the graph above.

   (a) Find $\int_0^{24} v(t)\,dt$. Using correct units, explain the meaning of $\int_0^{24} v(t)\,dt$.

   (b) For each of $v'(4)$ and $v'(20)$, find the value or explain why it does not exist. Indicate units of measure.

   (c) Let $a(t)$ be the car's acceleration at time t, in meters per second per second. For $0 < t < 24$, write a piecewise-defined function for $a(t)$.

   (d) Find the average rate of change of v over the interval $8 \le t \le 20$. Does the Mean Value Theorem guarantee a value of c, for $8 < c < 20$, such that $v'(c)$ is equal to this average rate of change? Why or why not?

---

6. Let f be a function with derivatives of all orders and for which $f(2) = 7$. When n is odd, the nth derivative of f at $x = 2$ is 0. When n is even and $n \ge 2$, the nth derivative of f at $x = 2$ is given by $f^{(n)}(2) = \dfrac{(n-1)!}{3^n}$.

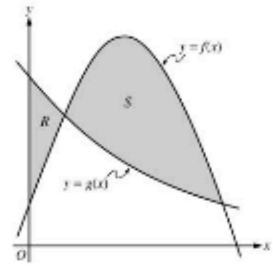
   (a) Write the sixth-degree Taylor polynomial for f about $x = 2$.

   (b) In the Taylor series for f about $x = 2$, what is the coefficient of $(x-2)^{2n}$ for $n \ge 1$?

   (c) Find the interval of convergence of the Taylor series for f about $x = 2$. Show the work that leads to your answer.

Question 1

Let f and g be the functions given by $f(x) = \frac{1}{4} + \sin(\pi x)$ and $g(x) = 4^{-x}$. Let R be the shaded region in the first quadrant enclosed by the y-axis and the graphs of f and g, and let S be the shaded region in the first quadrant enclosed by the graphs of f and g, as shown in the figure above.



(a) Find the area of R.

(b) Find the area of S.

(c) Find the volume of the solid generated when S is revolved about the horizontal line $y = -1$.

$f(x) = g(x)$ when $\frac{1}{4} + \sin(\pi x) = 4^{-x}$.

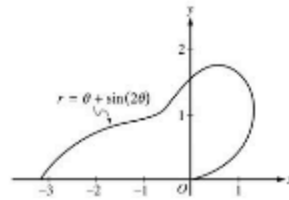
f and g intersect when $x = 0.178218$ and when $x = 1$.

Let $a = 0.178218$.

(a) $\int_0^a (g(x) - f(x))\,dx = 0.064$ or 0.065	3 : { 1 : limits 1 : integrand 1 : answer
(b) $\int_a^1 (f(x) - g(x))\,dx = 0.410$	3 : { 1 : limits 1 : integrand 1 : answer
(c) $\pi\int_a^1 \left((f(x)+1)^2 - (g(x)+1)^2\right)dx = 4.558$ or 4.559	3 : { 2 : integrand 1 : limits, constant, and answer

Question 2

The curve above is drawn in the xy-plane and is described by the equation in polar coordinates $r = \theta + \sin(2\theta)$ for $0 \le \theta \le \pi$, where r is measured in meters and θ is measured in radians. The derivative of r with respect to θ is given by $\frac{dr}{d\theta} = 1 + 2\cos(2\theta)$.



(a) Find the area bounded by the curve and the x-axis.

(b) Find the angle θ that corresponds to the point on the curve with x-coordinate -2.

(c) For $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$, $\frac{dr}{d\theta}$ is negative. What does this fact say about r? What does this fact say about the curve?

(d) Find the value of θ in the interval $0 \le \theta \le \frac{\pi}{2}$ that corresponds to the point on the curve in the first quadrant with greatest distance from the origin. Justify your answer.

(a) Area $= \frac{1}{2}\int_0^{\pi} r^2\, d\theta$

$$= \frac{1}{2}\int_0^{\pi} (\theta + \sin(2\theta))^2\, d\theta = 4.382$$

$3: \begin{cases} 1: \text{limits and constant} \\ 1: \text{integrand} \\ 1: \text{answer} \end{cases}$

(b) $-2 = r\cos(\theta) = (\theta + \sin(2\theta))\cos(\theta)$
$\theta = 2.786$

$2: \begin{cases} 1: \text{equation} \\ 1: \text{answer} \end{cases}$

(c) Since $\frac{dr}{d\theta} < 0$ for $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$, r is decreasing on this interval. This means the curve is getting closer to the origin.

$2: \begin{cases} 1: \text{information about } r \\ 1: \text{information about the curve} \end{cases}$

(d) The only value in $\left[0, \frac{\pi}{2}\right]$ where $\frac{dr}{d\theta} = 0$ is $\theta = \frac{\pi}{3}$.

θ	r
0	0
$\frac{\pi}{3}$	1.913
$\frac{\pi}{2}$	1.571

The greatest distance occurs when $\theta = \frac{\pi}{3}$.

$2: \begin{cases} 1: \theta = \frac{\pi}{3} \text{ or } 1.047 \\ 1: \text{answer with justification} \end{cases}$

Question 3

Distance x (cm)	0	1	5	6	8
Temperature $T(x)$ (°C)	100	93	70	62	55

A metal wire of length 8 centimeters (cm) is heated at one end. The table above gives selected values of the temperature $T(x)$, in degrees Celsius (°C), of the wire x cm from the heated end. The function T is decreasing and twice differentiable.

(a) Estimate $T'(7)$. Show the work that leads to your answer. Indicate units of measure.

(b) Write an integral expression in terms of $T(x)$ for the average temperature of the wire. Estimate the average temperature of the wire using a trapezoidal sum with the four subintervals indicated by the data in the table. Indicate units of measure.

(c) Find $\int_0^8 T'(x)\,dx$, and indicate units of measure. Explain the meaning of $\int_0^8 T'(x)\,dx$ in terms of the temperature of the wire.

(d) Are the data in the table consistent with the assertion that $T''(x) > 0$ for every x in the interval $0 < x < 8$? Explain your answer.

(a) $\dfrac{T(8)-T(6)}{8-6} = \dfrac{55-62}{2} = -\dfrac{7}{2}$ °C/cm

1 : answer

(b) $\frac{1}{8}\int_0^8 T(x)\,dx$

Trapezoidal approximation for $\int_0^8 T(x)\,dx$:

$A = \frac{100+93}{2}\cdot 1 + \frac{93+70}{2}\cdot 4 + \frac{70+62}{2}\cdot 1 + \frac{62+55}{2}\cdot 2$

Average temperature $\approx \frac{1}{8}A = 75.6875$ °C

3 : $\begin{cases} 1 : \frac{1}{8}\int_0^8 T(x)\,dx \\ 1 : \text{trapezoidal sum} \\ 1 : \text{answer} \end{cases}$

(c) $\int_0^8 T'(x)\,dx = T(8) - T(0) = 55 - 100 = -45$ °C

The temperature drops 45°C from the heated end of the wire to the other end of the wire.

2 : $\begin{cases} 1 : \text{value} \\ 1 : \text{meaning} \end{cases}$

(d) Average rate of change of temperature on $[1, 5]$ is $\frac{70-93}{5-1} = -5.75$.

Average rate of change of temperature on $[5, 6]$ is $\frac{62-70}{6-5} = -8$.

No. By the MVT, $T'(c_1) = -5.75$ for some c_1 in the interval $(1, 5)$ and $T'(c_2) = -8$ for some c_2 in the interval $(5, 6)$. It follows that T' must decrease somewhere in the interval (c_1, c_2). Therefore T'' is not positive for every x in $[0, 8]$.

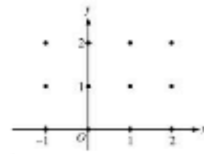
2 : $\begin{cases} 1 : \text{two slopes of secant lines} \\ 1 : \text{answer with explanation} \end{cases}$

Units of °C/cm in (a), and °C in (b) and (c)

1 : units in (a), (b), and (c)

Question 4

Consider the differential equation $\frac{dy}{dx} = 2x - y$.



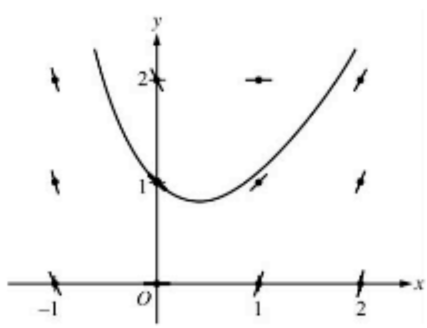
(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated, and sketch the solution curve that passes through the point $(0, 1)$. **(Note: Use the axes provided in the pink test booklet.)**

(b) The solution curve that passes through the point $(0, 1)$ has a local minimum at $x = \ln\left(\frac{3}{2}\right)$. What is the y-coordinate of this local minimum?

(c) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(0) = 1$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f(-0.4)$. Show the work that leads to your answer.

(d) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine whether the approximation found in part (c) is less than or greater than $f(-0.4)$. Explain your reasoning.

(a)



$3: \begin{cases} 1: \text{zero slopes} \\ 1: \text{nonzero slopes} \\ 1: \text{curve through } (0, 1) \end{cases}$

(b) $\frac{dy}{dx} = 0$ when $2x = y$

The y-coordinate is $2\ln\left(\frac{3}{2}\right)$.

$2: \begin{cases} 1: \text{sets } \frac{dy}{dx} = 0 \\ 1: \text{answer} \end{cases}$

(c) $f(-0.2) \approx f(0) + f'(0)(-0.2)$
$= 1 + (-1)(-0.2) = 1.2$
$f(-0.4) \approx f(-0.2) + f'(-0.2)(-0.2)$
$\approx 1.2 + (-1.6)(-0.2) = 1.52$

$2: \begin{cases} 1: \text{Euler's method with two steps} \\ 1: \text{Euler approximation to } f(-0.4) \end{cases}$

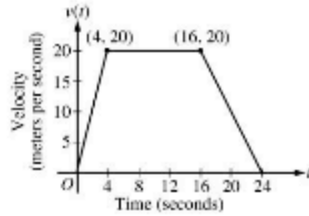
(d) $\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx} = 2 - 2x + y$

$\frac{d^2y}{dx^2}$ is positive in quadrant II because $x < 0$ and $y > 0$.
$1.52 < f(-0.4)$ since all solution curves in quadrant II are concave up.

$2: \begin{cases} 1: \frac{d^2y}{dx^2} \\ 1: \text{answer with reason} \end{cases}$

Question 5

A car is traveling on a straight road. For $0 \le t \le 24$ seconds, the car's velocity $v(t)$, in meters per second, is modeled by the piecewise-linear function defined by the graph above.



(a) Find $\int_0^{24} v(t)\,dt$. Using correct units, explain the meaning of $\int_0^{24} v(t)\,dt$.

(b) For each of $v'(4)$ and $v'(20)$, find the value or explain why it does not exist. Indicate units of measure.

(c) Let $a(t)$ be the car's acceleration at time t, in meters per second per second. For $0 < t < 24$, write a piecewise-defined function for $a(t)$.

(d) Find the average rate of change of v over the interval $8 \le t \le 20$. Does the Mean Value Theorem guarantee a value of c, for $8 < c < 20$, such that $v'(c)$ is equal to this average rate of change? Why or why not?

(a) $\int_0^{24} v(t)\,dt = \frac{1}{2}(4)(20) + (12)(20) + \frac{1}{2}(8)(20) = 360$

The car travels 360 meters in these 24 seconds.

$2: \begin{cases} 1: \text{value} \\ 1: \text{meaning with units} \end{cases}$

(b) $v'(4)$ does not exist because

$$\lim_{t \to 4^-}\left(\frac{v(t)-v(4)}{t-4}\right) = 5 \ne 0 = \lim_{t \to 4^+}\left(\frac{v(t)-v(4)}{t-4}\right).$$

$v'(20) = \frac{20-0}{16-24} = -\frac{5}{2}$ m/sec^2

$3: \begin{cases} 1: v'(4) \text{ does not exist, with explanation} \\ 1: v'(20) \\ 1: \text{units} \end{cases}$

(c)

$$a(t) = \begin{cases} 5 & \text{if } 0 < t < 4 \\ 0 & \text{if } 4 < t < 16 \\ -\frac{5}{2} & \text{if } 16 < t < 24 \end{cases}$$

$a(t)$ does not exist at $t = 4$ and $t = 16$.

$2: \begin{cases} 1: \text{finds the values } 5, 0, -\frac{5}{2} \\ 1: \text{identifies constants with correct intervals} \end{cases}$

(d) The average rate of change of v on $[8, 20]$ is

$\frac{v(20)-v(8)}{20-8} = -\frac{5}{6}$ m/sec^2.

No, the Mean Value Theorem does not apply to v on $[8, 20]$ because v is not differentiable at $t = 16$.

$2: \begin{cases} 1: \text{average rate of change of } v \text{ on } [8, 20] \\ 1: \text{answer with explanation} \end{cases}$

Question 6

Let f be a function with derivatives of all orders and for which $f(2) = 7$. When n is odd, the nth derivative of f at $x = 2$ is 0. When n is even and $n \geq 2$, the nth derivative of f at $x = 2$ is given by $f^{(n)}(2) = \frac{(n-1)!}{3^n}$.

(a) Write the sixth-degree Taylor polynomial for f about $x = 2$.

(b) In the Taylor series for f about $x = 2$, what is the coefficient of $(x-2)^{2n}$ for $n \geq 1$?

(c) Find the interval of convergence of the Taylor series for f about $x = 2$. Show the work that leads to your answer.

(a) $P_6(x) = 7 + \frac{1!}{3^2} \cdot \frac{1}{2!}(x-2)^2 + \frac{3!}{3^4} \cdot \frac{1}{4!}(x-2)^4 + \frac{5!}{3^6} \cdot \frac{1}{6!}(x-2)^6$

3 :
- 1 : polynomial about $x = 2$
- 2 : $P_6(x)$
  - $\langle -1 \rangle$ each incorrect term
  - $\langle -1 \rangle$ max for all extra terms, $+\cdots$, misuse of equality

(b) $\frac{(2n-1)!}{3^{2n}} \cdot \frac{1}{(2n)!} = \frac{1}{3^{2n}(2n)}$

1 : coefficient

(c) The Taylor series for f about $x = 2$ is

$$f(x) = 7 + \sum_{n=1}^{\infty} \frac{1}{2n \cdot 3^{2n}}(x-2)^{2n}.$$

$$L = \lim_{n \to \infty} \left| \frac{\frac{1}{2(n+1)} \cdot \frac{1}{3^{2(n+1)}}(x-2)^{2(n+1)}}{\frac{1}{2n} \cdot \frac{1}{3^{2n}}(x-2)^{2n}} \right|$$

$$= \lim_{n \to \infty} \left| \frac{2n}{2(n+1)} \cdot \frac{3^{2n}}{3^2 3^{2n}}(x-2)^2 \right| = \frac{(x-2)^2}{9}$$

$L < 1$ when $|x-2| < 3$.

Thus, the series converges when $-1 < x < 5$.

When $x = 5$, the series is $7 + \sum_{n=1}^{\infty} \frac{3^{2n}}{2n \cdot 3^{2n}} = 7 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$,

which diverges, because $\sum_{n=1}^{\infty} \frac{1}{n}$, the harmonic series, diverges.

When $x = -1$, the series is $7 + \sum_{n=1}^{\infty} \frac{(-3)^{2n}}{2n \cdot 3^{2n}} = 7 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$,

which diverges, because $\sum_{n=1}^{\infty} \frac{1}{n}$, the harmonic series, diverges.

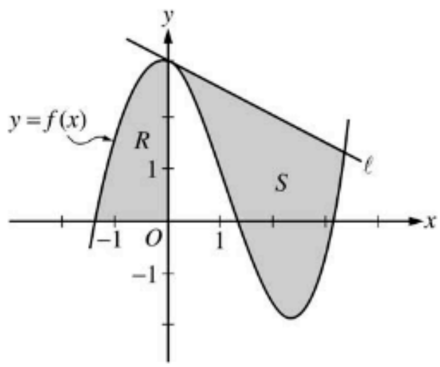
The interval of convergence is $(-1, 5)$.

5 :
- 1 : sets up ratio
- 1 : computes limit of ratio
- 1 : identifies interior of interval of convergence
- 1 : considers both endpoints
- 1 : analysis/conclusion for both endpoints

2006 B

CALCULUS BC
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.



1. Let f be the function given by $f(x) = \frac{x^3}{4} - \frac{x^2}{3} - \frac{x}{2} + 3\cos x$. Let R be the shaded region in the second quadrant bounded by the graph of f, and let S be the shaded region bounded by the graph of f and line ℓ, the line tangent to the graph of f at $x = 0$, as shown above.

   (a) Find the area of R.

   (b) Find the volume of the solid generated when R is rotated about the horizontal line $y = -2$.

   (c) Write, but do not evaluate, an integral expression that can be used to find the area of S.

---

2. An object moving along a curve in the xy-plane is at position $(x(t), y(t))$ at time t, where

$$\frac{dx}{dt} = \tan\left(e^{-t}\right) \text{ and } \frac{dy}{dt} = \sec\left(e^{-t}\right)$$

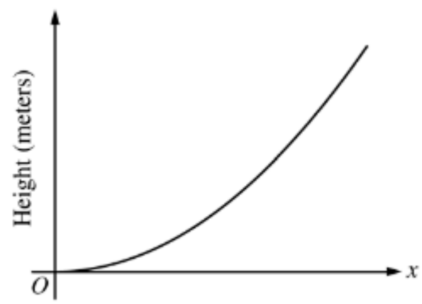
   for $t \geq 0$. At time $t = 1$, the object is at position $(2, -3)$.

   (a) Write an equation for the line tangent to the curve at position $(2, -3)$.

   (b) Find the acceleration vector and the speed of the object at time $t = 1$.

   (c) Find the total distance traveled by the object over the time interval $1 \leq t \leq 2$.

   (d) Is there a time $t \geq 0$ at which the object is on the y-axis? Explain why or why not.



3. The figure above is the graph of a function of x, which models the height of a skateboard ramp. The function meets the following requirements.

   (i) At $x = 0$, the value of the function is 0, and the slope of the graph of the function is 0.

   (ii) At $x = 4$, the value of the function is 1, and the slope of the graph of the function is 1.

   (iii) Between $x = 0$ and $x = 4$, the function is increasing.

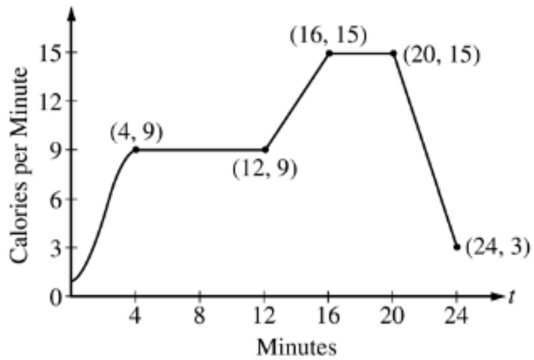
   (a) Let $f(x) = ax^2$, where a is a nonzero constant. Show that it is not possible to find a value for a so that f meets requirement (ii) above.

   (b) Let $g(x) = cx^3 - \frac{x^2}{16}$, where c is a nonzero constant. Find the value of c so that g meets requirement (ii) above. Show the work that leads to your answer.

   (c) Using the function g and your value of c from part (b), show that g does not meet requirement (iii) above.

   (d) Let $h(x) = \frac{x^n}{k}$, where k is a nonzero constant and n is a positive integer. Find the values of k and n so that h meets requirement (ii) above. Show that h also meets requirements (i) and (iii) above.

No calculator is allowed for these problems.



4. The rate, in calories per minute, at which a person using an exercise machine burns calories is modeled by the function f. In the figure above, $f(t) = -\frac{1}{4}t^3 + \frac{3}{2}t^2 + 1$ for $0 \le t \le 4$ and f is piecewise linear for $4 \le t \le 24$.

   (a) Find $f'(22)$. Indicate units of measure.

   (b) For the time interval $0 \le t \le 24$, at what time t is f increasing at its greatest rate? Show the reasoning that supports your answer.

   (c) Find the total number of calories burned over the time interval $6 \le t \le 18$ minutes.

   (d) The setting on the machine is now changed so that the person burns $f(t) + c$ calories per minute. For this setting, find c so that an average of 15 calories per minute is burned during the time interval $6 \le t \le 18$.

---

5. Let f be a function with $f(4) = 1$ such that all points (x, y) on the graph of f satisfy the differential equation

$$\frac{dy}{dx} = 2y(3 - x).$$

   Let g be a function with $g(4) = 1$ such that all points (x, y) on the graph of g satisfy the logistic differential equation

$$\frac{dy}{dx} = 2y(3 - y).$$

   (a) Find $y = f(x)$.

   (b) Given that $g(4) = 1$, find $\lim_{x \to \infty} g(x)$ and $\lim_{x \to \infty} g'(x)$. (It is not necessary to solve for $g(x)$ or to show how you arrived at your answers.)

   (c) For what value of y does the graph of g have a point of inflection? Find the slope of the graph of g at the point of inflection. (It is not necessary to solve for $g(x)$.)

---

6. The function f is defined by $f(x) = \frac{1}{1 + x^3}$. The Maclaurin series for f is given by

$$1 - x^3 + x^6 - x^9 + \cdots + (-1)^n x^{3n} + \cdots,$$

which converges to $f(x)$ for $-1 < x < 1$.

(a) Find the first three nonzero terms and the general term for the Maclaurin series for $f'(x)$.

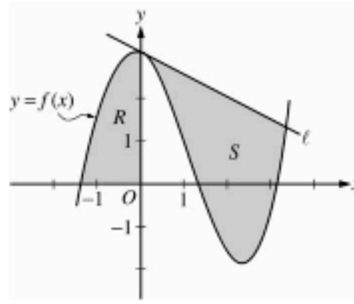
(b) Use your results from part (a) to find the sum of the infinite series $-\frac{3}{2^2} + \frac{6}{2^5} - \frac{9}{2^8} + \cdots + (-1)^n \frac{3n}{2^{3n-1}} + \cdots$.

(c) Find the first four nonzero terms and the general term for the Maclaurin series representing $\int_0^x f(t)\,dt$.

(d) Use the first three nonzero terms of the infinite series found in part (c) to approximate $\int_0^{1/2} f(t)\,dt$. What are the properties of the terms of the series representing $\int_0^{1/2} f(t)\,dt$ that guarantee that this approximation is within $\frac{1}{10{,}000}$ of the exact value of the integral?

Question 1

Let f be the function given by $f(x) = \frac{x^3}{4} - \frac{x^2}{3} - \frac{x}{2} + 3\cos x$. Let R be the shaded region in the second quadrant bounded by the graph of f, and let S be the shaded region bounded by the graph of f and line ℓ, the line tangent to the graph of f at $x = 0$, as shown above.



(a) Find the area of R.

(b) Find the volume of the solid generated when R is rotated about the horizontal line $y = -2$.

(c) Write, but do not evaluate, an integral expression that can be used to find the area of S.

For $x < 0$, $f(x) = 0$ when $x = -1.37312$.
Let $P = -1.37312$.

(a) Area of $R = \int_P^0 f(x)\,dx = 2.903$

2 : { 1 : integral; 1 : answer }

(b) Volume $= \pi\int_P^0 \left((f(x)+2)^2 - 4\right) dx = 59.361$

4 : { 1 : limits and constant; 2 : integrand; 1 : answer }

(c) The equation of the tangent line ℓ is $y = 3 - \frac{1}{2}x$.

The graph of f and line ℓ intersect at $A = 3.38987$.

Area of $S = \int_0^A \left(\left(3 - \frac{1}{2}x\right) - f(x)\right) dx$

3 : { 1 : tangent line; 1 : integrand; 1 : limits }

Question 2

An object moving along a curve in the xy-plane is at position $(x(t), y(t))$ at time t, where

$$\frac{dx}{dt} = \tan\left(e^{-t}\right) \text{ and } \frac{dy}{dt} = \sec\left(e^{-t}\right)$$

for $t \geq 0$. At time $t = 1$, the object is at position $(2, -3)$.

(a) Write an equation for the line tangent to the curve at position $(2, -3)$.

(b) Find the acceleration vector and the speed of the object at time $t = 1$.

(c) Find the total distance traveled by the object over the time interval $1 \leq t \leq 2$.

(d) Is there a time $t \geq 0$ at which the object is on the y-axis? Explain why or why not.

(a) $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{\sec\left(e^{-t}\right)}{\tan\left(e^{-t}\right)} = \dfrac{1}{\sin\left(e^{-t}\right)}$

$\left.\dfrac{dy}{dx}\right|_{(2,-3)} = \dfrac{1}{\sin\left(e^{-1}\right)} = 2.780$ or 2.781

$y + 3 = \dfrac{1}{\sin\left(e^{-1}\right)}(x - 2)$

$2: \begin{cases} 1: \left.\dfrac{dy}{dx}\right|_{(2,-3)} \\ 1: \text{equation of tangent line} \end{cases}$

(b) $x''(1) = -0.42253,\ y''(1) = -0.15196$

$a(1) = \langle -0.423, -0.152 \rangle$ or $\langle -0.422, -0.151 \rangle$.

speed $= \sqrt{\left(\sec\left(e^{-1}\right)\right)^2 + \left(\tan\left(e^{-1}\right)\right)^2} = 1.138$ or 1.139

$2: \begin{cases} 1: \text{acceleration vector} \\ 1: \text{speed} \end{cases}$

(c) $\displaystyle\int_1^2 \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\, dt = 1.059$

$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$

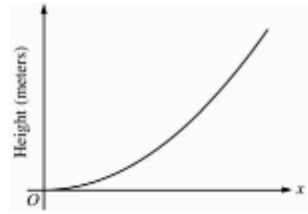
(d) $x(0) = x(1) - \displaystyle\int_0^1 x'(t)\, dt = 2 - 0.775553 > 0$

The particle starts to the right of the y-axis.
Since $x'(t) > 0$ for all $t \geq 0$, the object is always moving to the right and thus is never on the y-axis.

$3: \begin{cases} 1: x(0) \text{ expression} \\ 1: x'(t) > 0 \\ 1: \text{conclusion and reason} \end{cases}$

Question 3

The figure above is the graph of a function of x, which models the height of a skateboard ramp. The function meets the following requirements.



(i) At $x = 0$, the value of the function is 0, and the slope of the graph of the function is 0.
(ii) At $x = 4$, the value of the function is 1, and the slope of the graph of the function is 1.
(iii) Between $x = 0$ and $x = 4$, the function is increasing.

(a) Let $f(x) = ax^2$, where a is a nonzero constant. Show that it is not possible to find a value for a so that f meets requirement (ii) above.

(b) Let $g(x) = cx^3 - \frac{x^2}{16}$, where c is a nonzero constant. Find the value of c so that g meets requirement (ii) above. Show the work that leads to your answer.

(c) Using the function g and your value of c from part (b), show that g does not meet requirement (iii) above.

(d) Let $h(x) = \frac{x^n}{k}$, where k is a nonzero constant and n is a positive integer. Find the values of k and n so that h meets requirement (ii) above. Show that h also meets requirements (i) and (iii) above.

(a) $f(4) = 1$ implies that $a = \frac{1}{16}$ and $f'(4) = 2a(4) = 1$ implies that $a = \frac{1}{8}$. Thus, f cannot satisfy (ii).

$2: \begin{cases} 1: a = \frac{1}{16} \text{ or } a = \frac{1}{8} \\ 1: \text{shows } a \text{ does not work} \end{cases}$

(b) $g(4) = 64c - 1 = 1$ implies that $c = \frac{1}{32}$.

When $c = \frac{1}{32}$, $g'(4) = 3c(4)^2 - \frac{2(4)}{16} = 3\left(\frac{1}{32}\right)(16) - \frac{1}{2} = 1$

$1:$ value of c

(c) $g'(x) = \frac{3}{32}x^2 - \frac{x}{8} = \frac{1}{32}x(3x - 4)$

$g'(x) < 0$ for $0 < x < \frac{4}{3}$, so g does not satisfy (iii).

$2: \begin{cases} 1: g'(x) \\ 1: \text{explanation} \end{cases}$

(d) $h(4) = \frac{4^n}{k} = 1$ implies that $4^n = k$.

$h'(4) = \frac{n4^{n-1}}{k} = \frac{n4^{n-1}}{4^n} = \frac{n}{4} = 1$ gives $n = 4$ and $k = 4^4 = 256$.

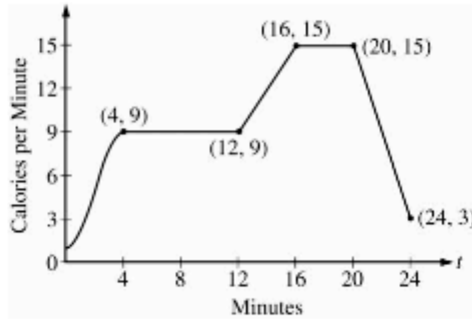
$h(x) = \frac{x^4}{256} \Rightarrow h(0) = 0$.

$h'(x) = \frac{4x^3}{256} \Rightarrow h'(0) = 0$ and $h'(x) > 0$ for $0 < x < 4$.

$4: \begin{cases} 1: \frac{4^n}{k} = 1 \\ 1: \frac{n4^{n-1}}{k} = 1 \\ 1: \text{values for } k \text{ and } n \\ 1: \text{verifications} \end{cases}$

Question 4

The rate, in calories per minute, at which a person using an exercise machine burns calories is modeled by the function f. In the figure above, $f(t) = -\frac{1}{4}t^3 + \frac{3}{2}t^2 + 1$ for $0 \le t \le 4$ and f is piecewise linear for $4 \le t \le 24$.



(a) Find $f'(22)$. Indicate units of measure.

(b) For the time interval $0 \le t \le 24$, at what time t is f increasing at its greatest rate? Show the reasoning that supports your answer.

(c) Find the total number of calories burned over the time interval $6 \le t \le 18$ minutes.

(d) The setting on the machine is now changed so that the person burns $f(t) + c$ calories per minute. For this setting, find c so that an average of 15 calories per minute is burned during the time interval $6 \le t \le 18$.

(a) $f'(22) = \frac{15-3}{20-24} = -3$ calories/min/min

1 : $f'(22)$ and units

(b) f is increasing on $[0, 4]$ and on $[12, 16]$.

On $(12, 16)$, $f'(t) = \frac{15-9}{16-12} = \frac{3}{2}$ since f has constant slope on this interval.

On $(0, 4)$, $f'(t) = -\frac{3}{4}t^2 + 3t$ and

$f''(t) = -\frac{3}{2}t + 3 = 0$ when $t = 2$. This is where f' has a maximum on $[0, 4]$ since $f'' > 0$ on $(0, 2)$ and $f'' < 0$ on $(2, 4)$.

On $[0, 24]$, f is increasing at its greatest rate when $t = 2$ because $f'(2) = 3 > \frac{3}{2}$.

4 : $\begin{cases} 1 : f' \text{ on } (0, 4) \\ 1 : \text{shows } f' \text{ has a max at } t = 2 \text{ on } (0, 4) \\ 1 : \text{shows for } 12 < t < 16,\ f'(t) < f'(2) \\ 1 : \text{answer} \end{cases}$

(c) $\int_6^{18} f(t)\,dt = 6(9) + \frac{1}{2}(4)(9 + 15) + 2(15)$

$= 132$ calories

2 : $\begin{cases} 1 : \text{method} \\ 1 : \text{answer} \end{cases}$

(d) We want $\frac{1}{12}\int_6^{18} (f(t) + c)\,dt = 15$.

This means $132 + 12c = 15(12)$. So, $c = 4$.

OR

Currently, the average is $\frac{132}{12} = 11$ calories/min.

Adding c to $f(t)$ will shift the average by c.

So $c = 4$ to get an average of 15 calories/min.

2 : $\begin{cases} 1 : \text{setup} \\ 1 : \text{value of } c \end{cases}$

Question 5

Let f be a function with $f(4) = 1$ such that all points (x, y) on the graph of f satisfy the differential equation

$$\frac{dy}{dx} = 2y(3 - x).$$

Let g be a function with $g(4) = 1$ such that all points (x, y) on the graph of g satisfy the logistic differential equation

$$\frac{dy}{dx} = 2y(3 - y).$$

(a) Find $y = f(x)$.

(b) Given that $g(4) = 1$, find $\lim_{x \to \infty} g(x)$ and $\lim_{x \to \infty} g'(x)$. (It is not necessary to solve for $g(x)$ or to show how you arrived at your answers.)

(c) For what value of y does the graph of g have a point of inflection? Find the slope of the graph of g at the point of inflection. (It is not necessary to solve for $g(x)$.)

(a) $\frac{dy}{dx} = 2y(3 - x)$

$\frac{1}{y}\,dy = 2(3 - x)\,dx$

$\ln|y| = 6x - x^2 + C$

$0 = 24 - 16 + C$

$C = -8$

$\ln|y| = 6x - x^2 - 8$

$y = e^{6x - x^2 - 8}$ for $-\infty < x < \infty$

$5: \begin{cases} 1: \text{separates variables} \\ 1: \text{antiderivatives} \\ 1: \text{constant of integration} \\ 1: \text{uses initial condition} \\ 1: \text{solution} \end{cases}$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

(b) $\lim_{x \to \infty} g(x) = 3$

$\lim_{x \to \infty} g'(x) = 0$

$2: \begin{cases} 1: \lim_{x \to \infty} g(x) = 3 \\ 1: \lim_{x \to \infty} g'(x) = 0 \end{cases}$

(c) $\frac{d^2y}{dx^2} = (6 - 4y)\frac{dy}{dx}$

Because $\frac{dy}{dx} \neq 0$ at any point on the graph of g, the concavity only changes sign at $y = \frac{3}{2}$, half the carrying capacity.

$\left.\frac{dy}{dx}\right|_{y=3/2} = 2\left(\frac{3}{2}\right)\left(3 - \frac{3}{2}\right) = \frac{9}{2}$

$2: \begin{cases} 1: y = \frac{3}{2} \\ 1: \left.\frac{dy}{dx}\right|_{y=3/2} \end{cases}$

Question 6

The function f is defined by $f(x) = \dfrac{1}{1+x^3}$. The Maclaurin series for f is given by

$$1 - x^3 + x^6 - x^9 + \cdots + (-1)^n x^{3n} + \cdots,$$

which converges to $f(x)$ for $-1 < x < 1$.

(a) Find the first three nonzero terms and the general term for the Maclaurin series for $f'(x)$.

(b) Use your results from part (a) to find the sum of the infinite series $-\dfrac{3}{2^2} + \dfrac{6}{2^5} - \dfrac{9}{2^8} + \cdots + (-1)^n \dfrac{3n}{2^{3n-1}} + \cdots$.

(c) Find the first four nonzero terms and the general term for the Maclaurin series representing $\int_0^x f(t)\,dt$.

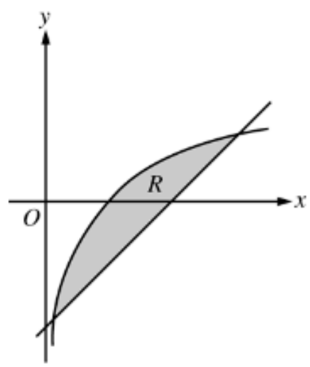
(d) Use the first three nonzero terms of the infinite series found in part (c) to approximate $\int_0^{1/2} f(t)\,dt$. What are the properties of the terms of the series representing $\int_0^{1/2} f(t)\,dt$ that guarantee that this approximation is within $\dfrac{1}{10{,}000}$ of the exact value of the integral?

Solution	Scoring
(a) $f'(x) = -3x^2 + 6x^5 - 9x^8 + \cdots + 3n(-1)^n x^{3n-1} + \cdots$	$2: \begin{cases} 1: \text{first three terms} \\ 1: \text{general term} \end{cases}$
(b) The given series is the Maclaurin series for $f'(x)$ with $x = \frac{1}{2}$. $f'(x) = -\left(1+x^3\right)^{-2}\left(3x^2\right)$ Thus, the sum of the series is $f'\left(\frac{1}{2}\right) = -\dfrac{3\left(\frac{1}{4}\right)}{\left(1+\frac{1}{8}\right)^2} = -\dfrac{16}{27}$.	$2: \begin{cases} 1: f'(x) \\ 1: f'\left(\frac{1}{2}\right) \end{cases}$
(c) $\int_0^x \dfrac{1}{1+t^3}\,dt = x - \dfrac{x^4}{4} + \dfrac{x^7}{7} - \dfrac{x^{10}}{10} + \cdots + \dfrac{(-1)^n x^{3n+1}}{3n+1} + \cdots$	$2: \begin{cases} 1: \text{first four terms} \\ 1: \text{general term} \end{cases}$
(d) $\int_0^{1/2} \dfrac{1}{1+t^3}\,dt \approx \dfrac{1}{2} - \dfrac{\left(\frac{1}{2}\right)^4}{4} + \dfrac{\left(\frac{1}{2}\right)^7}{7}$. The series in part (c) with $x = \frac{1}{2}$ has terms that alternate, decrease in absolute value, and have limit 0. Hence the error is bounded by the absolute value of the next term. $\left\lvert \int_0^{1/2} \dfrac{1}{1+t^3}\,dt - \left(\dfrac{1}{2} - \dfrac{\left(\frac{1}{2}\right)^4}{4} + \dfrac{\left(\frac{1}{2}\right)^7}{7}\right) \right\rvert < \dfrac{\left(\frac{1}{2}\right)^{10}}{10} = \dfrac{1}{10240} < 0.0001$	$3: \begin{cases} 1: \text{approximation} \\ 1: \text{properties of terms} \\ 1: \text{absolute value of fourth term} < 0.0001 \end{cases}$

2006 AP® CALCULUS BC FREE-RESPONSE QUESTIONS

CALCULUS BC
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.

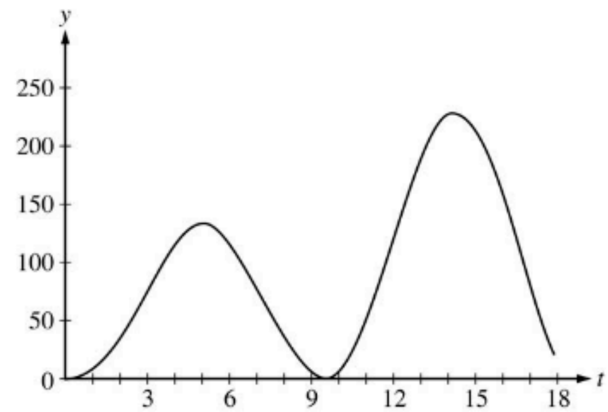


1. Let R be the shaded region bounded by the graph of $y = \ln x$ and the line $y = x - 2$, as shown above.

   (a) Find the area of R.

   (b) Find the volume of the solid generated when R is rotated about the horizontal line $y = -3$.

   (c) Write, but do not evaluate, an integral expression that can be used to find the volume of the solid generated when R is rotated about the y-axis.



2. At an intersection in Thomasville, Oregon, cars turn left at the rate $L(t) = 60\sqrt{t}\sin^2\left(\frac{t}{3}\right)$ cars per hour over the time interval $0 \le t \le 18$ hours. The graph of $y = L(t)$ is shown above.

(a) To the nearest whole number, find the total number of cars turning left at the intersection over the time interval $0 \le t \le 18$ hours.

(b) Traffic engineers will consider turn restrictions when $L(t) \ge 150$ cars per hour. Find all values of t for which $L(t) \ge 150$ and compute the average value of L over this time interval. Indicate units of measure.

(c) Traffic engineers will install a signal if there is any two-hour time interval during which the product of the total number of cars turning left and the total number of oncoming cars traveling straight through the intersection is greater than 200,000. In every two-hour time interval, 500 oncoming cars travel straight through the intersection. Does this intersection require a traffic signal? Explain the reasoning that leads to your conclusion.

---

3. An object moving along a curve in the xy-plane is at position $(x(t), y(t))$ at time t, where

$$\frac{dx}{dt} = \sin^{-1}\left(1 - 2e^{-t}\right) \text{ and } \frac{dy}{dt} = \frac{4t}{1 + t^3}$$

for $t \ge 0$. At time $t = 2$, the object is at the point $(6, -3)$. (Note: $\sin^{-1} x = \arcsin x$)

(a) Find the acceleration vector and the speed of the object at time $t = 2$.

(b) The curve has a vertical tangent line at one point. At what time t is the object at this point?

(c) Let $m(t)$ denote the slope of the line tangent to the curve at the point $(x(t), y(t))$. Write an expression for $m(t)$ in terms of t and use it to evaluate $\lim_{t \to \infty} m(t)$.

(d) The graph of the curve has a horizontal asymptote $y = c$. Write, but do not evaluate, an expression involving an improper integral that represents this value c.

No calculator is allowed for these problems.

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t (seconds)	0	10	20	30	40	50	60	70	80
$v(t)$ (feet per second)	5	14	22	29	35	40	44	47	49

4. Rocket A has positive velocity $v(t)$ after being launched upward from an initial height of 0 feet at time $t = 0$ seconds. The velocity of the rocket is recorded for selected values of t over the interval $0 \le t \le 80$ seconds, as shown in the table above.

   (a) Find the average acceleration of rocket A over the time interval $0 \le t \le 80$ seconds. Indicate units of measure.

   (b) Using correct units, explain the meaning of $\int_{10}^{70} v(t)\,dt$ in terms of the rocket's flight. Use a midpoint Riemann sum with 3 subintervals of equal length to approximate $\int_{10}^{70} v(t)\,dt$.

   (c) Rocket B is launched upward with an acceleration of $a(t) = \dfrac{3}{\sqrt{t+1}}$ feet per second per second. At time $t = 0$ seconds, the initial height of the rocket is 0 feet, and the initial velocity is 2 feet per second. Which of the two rockets is traveling faster at time $t = 80$ seconds? Explain your answer.

---

5. Consider the differential equation $\dfrac{dy}{dx} = 5x^2 - \dfrac{6}{y-2}$ for $y \ne 2$. Let $y = f(x)$ be the particular solution to this differential equation with the initial condition $f(-1) = -4$.

   (a) Evaluate $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ at $(-1, -4)$.

   (b) Is it possible for the x-axis to be tangent to the graph of f at some point? Explain why or why not.

   (c) Find the second-degree Taylor polynomial for f about $x = -1$.

   (d) Use Euler's method, starting at $x = -1$ with two steps of equal size, to approximate $f(0)$. Show the work that leads to your answer.

---

6. The function f is defined by the power series

$$f(x) = -\frac{x}{2} + \frac{2x^2}{3} - \frac{3x^3}{4} + \cdots + \frac{(-1)^n n x^n}{n+1} + \cdots$$

   for all real numbers x for which the series converges. The function g is defined by the power series

$$g(x) = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \cdots + \frac{(-1)^n x^n}{(2n)!} + \cdots$$

   for all real numbers x for which the series converges.

   (a) Find the interval of convergence of the power series for f. Justify your answer.

   (b) The graph of $y = f(x) - g(x)$ passes through the point $(0, -1)$. Find $y'(0)$ and $y''(0)$. Determine whether y has a relative minimum, a relative maximum, or neither at $x = 0$. Give a reason for your answer.

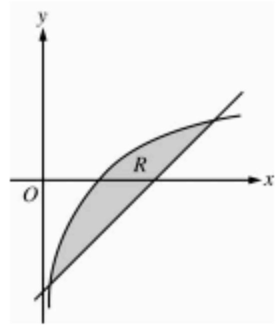
Question 1

Let R be the shaded region bounded by the graph of $y = \ln x$ and the line $y = x - 2$, as shown above.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is rotated about the horizontal line $y = -3$.

(c) Write, but do not evaluate, an integral expression that can be used to find the volume of the solid generated when R is rotated about the y-axis.



$\ln(x) = x - 2$ when $x = 0.15859$ and 3.14619.
Let $S = 0.15859$ and $T = 3.14619$

(a) Area of $R = \int_S^T (\ln(x) - (x-2))\,dx = 1.949$

$3: \begin{cases} 1: \text{integrand} \\ 1: \text{limits} \\ 1: \text{answer} \end{cases}$

(b) Volume $= \pi \int_S^T \left((\ln(x) + 3)^2 - (x - 2 + 3)^2 \right) dx$
$= 34.198$ or 34.199

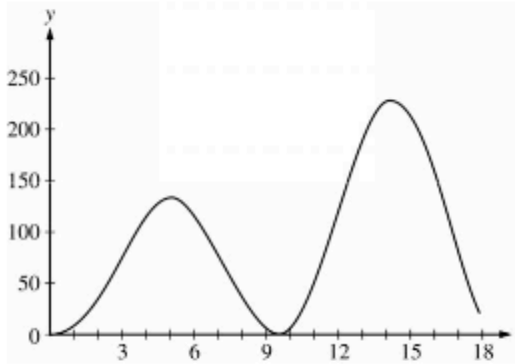
$3: \begin{cases} 2: \text{integrand} \\ 1: \text{limits, constant, and answer} \end{cases}$

(c) Volume $= \pi \int_{S-2}^{T-2} \left((y+2)^2 - \left(e^y\right)^2 \right) dy$

$3: \begin{cases} 2: \text{integrand} \\ 1: \text{limits and constant} \end{cases}$

Question 2

At an intersection in Thomasville, Oregon, cars turn left at the rate $L(t) = 60\sqrt{t}\sin^2\left(\frac{t}{3}\right)$ cars per hour over the time interval $0 \le t \le 18$ hours. The graph of $y = L(t)$ is shown above.



(a) To the nearest whole number, find the total number of cars turning left at the intersection over the time interval $0 \le t \le 18$ hours.

(b) Traffic engineers will consider turn restrictions when $L(t) \ge 150$ cars per hour. Find all values of t for which $L(t) \ge 150$ and compute the average value of L over this time interval. Indicate units of measure.

(c) Traffic engineers will install a signal if there is any two-hour time interval during which the product of the total number of cars turning left and the total number of oncoming cars traveling straight through the intersection is greater than 200,000. In every two-hour time interval, 500 oncoming cars travel straight through the intersection. Does this intersection require a traffic signal? Explain the reasoning that leads to your conclusion.

(a) $\int_0^{18} L(t)\,dt \approx 1658$ cars

$2: \begin{cases} 1: \text{setup} \\ 1: \text{answer} \end{cases}$

(b) $L(t) = 150$ when $t = 12.42831,\ 16.12166$

Let $R = 12.42831$ and $S = 16.12166$

$L(t) \ge 150$ for t in the interval $[R, S]$

$\frac{1}{S-R}\int_R^S L(t)\,dt = 199.426$ cars per hour

$3: \begin{cases} 1: t\text{-interval when } L(t) \ge 150 \\ 1: \text{average value integral} \\ 1: \text{answer with units} \end{cases}$

(c) For the product to exceed 200,000, the number of cars turning left in a two-hour interval must be greater than 400.

$\int_{13}^{15} L(t)\,dt = 431.931 > 400$

$4: \begin{cases} 1: \text{considers 400 cars} \\ 1: \text{valid interval } [h, h+2] \\ 1: \text{value of } \int_h^{h+2} L(t)\,dt \\ 1: \text{answer and explanation} \end{cases}$

OR

The number of cars turning left will be greater than 400 on a two-hour interval if $L(t) \ge 200$ on that interval.

$L(t) \ge 200$ on any two-hour subinterval of $[13.25304, 15.32386]$.

OR

$4: \begin{cases} 1: \text{considers 200 cars per hour} \\ 1: \text{solves } L(t) \ge 200 \\ 1: \text{discusses 2 hour interval} \\ 1: \text{answer and explanation} \end{cases}$

Yes, a traffic signal is required.

Question 3

An object moving along a curve in the xy-plane is at position $(x(t), y(t))$ at time t, where

$$\frac{dx}{dt} = \sin^{-1}\left(1 - 2e^{-t}\right) \text{ and } \frac{dy}{dt} = \frac{4t}{1+t^3}$$

for $t \geq 0$. At time $t = 2$, the object is at the point $(6, -3)$. (Note: $\sin^{-1} x = \arcsin x$)

(a) Find the acceleration vector and the speed of the object at time $t = 2$.

(b) The curve has a vertical tangent line at one point. At what time t is the object at this point?

(c) Let $m(t)$ denote the slope of the line tangent to the curve at the point $(x(t), y(t))$. Write an expression for $m(t)$ in terms of t and use it to evaluate $\lim_{t \to \infty} m(t)$.

(d) The graph of the curve has a horizontal asymptote $y = c$. Write, but do not evaluate, an expression involving an improper integral that represents this value c.

(a) $a(2) = \langle 0.395 \text{ or } 0.396, -0.741 \text{ or } -0.740 \rangle$

Speed $= \sqrt{x'(2)^2 + y'(2)^2} = 1.207$ or 1.208

$2 : \begin{cases} 1 : \text{acceleration} \\ 1 : \text{speed} \end{cases}$

(b) $\sin^{-1}\left(1 - 2e^{-t}\right) = 0$

$1 - 2e^{-t} = 0$

$t = \ln 2 = 0.693$ and $\frac{dy}{dt} \neq 0$ when $t = \ln 2$

$2 : \begin{cases} 1 : x'(t) = 0 \\ 1 : \text{answer} \end{cases}$

(c) $m(t) = \frac{4t}{1+t^3} \cdot \frac{1}{\sin^{-1}\left(1 - 2e^{-t}\right)}$

$$\lim_{t \to \infty} m(t) = \lim_{t \to \infty} \left(\frac{4t}{1+t^3} \cdot \frac{1}{\sin^{-1}\left(1 - 2e^{-t}\right)} \right)$$

$$= 0 \left(\frac{1}{\sin^{-1}(1)} \right) = 0$$

$2 : \begin{cases} 1 : m(t) \\ 1 : \text{limit value} \end{cases}$

(d) Since $\lim_{t \to \infty} x(t) = \infty$,

$$c = \lim_{t \to \infty} y(t) = -3 + \int_2^{\infty} \frac{4t}{1+t^3}\, dt$$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits} \\ 1 : \text{initial value consistent with lower limit} \end{cases}$

Question 4

t (seconds)	0	10	20	30	40	50	60	70	80
$v(t)$ (feet per second)	5	14	22	29	35	40	44	47	49

Rocket A has positive velocity $v(t)$ after being launched upward from an initial height of 0 feet at time $t = 0$ seconds. The velocity of the rocket is recorded for selected values of t over the interval $0 \le t \le 80$ seconds, as shown in the table above.

(a) Find the average acceleration of rocket A over the time interval $0 \le t \le 80$ seconds. Indicate units of measure.

(b) Using correct units, explain the meaning of $\int_{10}^{70} v(t)\,dt$ in terms of the rocket's flight. Use a midpoint Riemann sum with 3 subintervals of equal length to approximate $\int_{10}^{70} v(t)\,dt$.

(c) Rocket B is launched upward with an acceleration of $a(t) = \frac{3}{\sqrt{t+1}}$ feet per second per second. At time $t = 0$ seconds, the initial height of the rocket is 0 feet, and the initial velocity is 2 feet per second. Which of the two rockets is traveling faster at time $t = 80$ seconds? Explain your answer.

Solution	Scoring
(a) Average acceleration of rocket A is $\frac{v(80)-v(0)}{80-0} = \frac{49-5}{80} = \frac{11}{20}\ \text{ft}/\text{sec}^2$	1 : answer
(b) Since the velocity is positive, $\int_{10}^{70} v(t)\,dt$ represents the distance, in feet, traveled by rocket A from $t = 10$ seconds to $t = 70$ seconds. A midpoint Riemann sum is $20[v(20)+v(40)+v(60)]$ $= 20[22+35+44] = 2020$ ft	3 : { 1 : explanation 1 : uses $v(20)$, $v(40)$, $v(60)$ 1 : value
(c) Let $v_B(t)$ be the velocity of rocket B at time t. $v_B(t) = \int \frac{3}{\sqrt{t+1}}\,dt = 6\sqrt{t+1} + C$ $2 = v_B(0) = 6 + C$ $v_B(t) = 6\sqrt{t+1} - 4$ $v_B(80) = 50 > 49 = v(80)$ Rocket B is traveling faster at time $t = 80$ seconds.	4 : { 1 : $6\sqrt{t+1}$ 1 : constant of integration 1 : uses initial condition 1 : finds $v_B(80)$, compares to $v(80)$, and draws a conclusion
Units of ft/sec^2 in (a) and ft in (b)	1 : units in (a) and (b)

Question 5

Consider the differential equation $\frac{dy}{dx} = 5x^2 - \frac{6}{y-2}$ for $y \neq 2$. Let $y = f(x)$ be the particular solution to this differential equation with the initial condition $f(-1) = -4$.

(a) Evaluate $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $(-1, -4)$.

(b) Is it possible for the x-axis to be tangent to the graph of f at some point? Explain why or why not.

(c) Find the second-degree Taylor polynomial for f about $x = -1$.

(d) Use Euler's method, starting at $x = -1$ with two steps of equal size, to approximate $f(0)$. Show the work that leads to your answer.

Solution	Scoring
(a) $\left.\frac{dy}{dx}\right\|_{(-1,-4)} = 6$ $\frac{d^2y}{dx^2} = 10x + 6(y-2)^{-2}\frac{dy}{dx}$ $\left.\frac{d^2y}{dx^2}\right\|_{(-1,-4)} = -10 + 6\frac{1}{(-6)^2}6 = -9$	3 : { 1 : $\left.\frac{dy}{dx}\right\|_{(-1,-4)}$ 1 : $\frac{d^2y}{dx^2}$ 1 : $\left.\frac{d^2y}{dx^2}\right\|_{(-1,-4)}$
(b) The x-axis will be tangent to the graph of f if $\left.\frac{dy}{dx}\right\|_{(k,0)} = 0$. The x-axis will never be tangent to the graph of f because $\left.\frac{dy}{dx}\right\|_{(k,0)} = 5k^2 + 3 > 0$ for all k.	2 : { 1 : $\frac{dy}{dx} = 0$ and $y = 0$ 1 : answer and explanation
(c) $P(x) = -4 + 6(x+1) - \frac{9}{2}(x+1)^2$	2 : { 1 : quadratic and centered at $x = -1$ 1 : coefficients
(d) $f(-1) = -4$ $f\left(-\frac{1}{2}\right) \approx -4 + \frac{1}{2}(6) = -1$ $f(0) \approx -1 + \frac{1}{2}\left(\frac{5}{4} + 2\right) = \frac{5}{8}$	2 : { 1 : Euler's method with 2 steps 1 : Euler's approximation to $f(0)$

Question 6

The function f is defined by the power series

$$f(x) = -\frac{x}{2} + \frac{2x^2}{3} - \frac{3x^3}{4} + \cdots + \frac{(-1)^n nx^n}{n+1} + \cdots$$

for all real numbers x for which the series converges. The function g is defined by the power series

$$g(x) = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \cdots + \frac{(-1)^n x^n}{(2n)!} + \cdots$$

for all real numbers x for which the series converges.

(a) Find the interval of convergence of the power series for f. Justify your answer.

(b) The graph of $y = f(x) - g(x)$ passes through the point $(0, -1)$. Find $y'(0)$ and $y''(0)$. Determine whether y has a relative minimum, a relative maximum, or neither at $x = 0$. Give a reason for your answer.

(a) $\left|\frac{(-1)^{n+1}(n+1)x^{n+1}}{n+2} \cdot \frac{n+1}{(-1)^n nx^n}\right| = \frac{(n+1)^2}{(n+2)(n)} \cdot |x|$

$\lim_{n\to\infty} \frac{(n+1)^2}{(n+2)(n)} \cdot |x| = |x|$

The series converges when $-1 < x < 1$.

When $x = 1$, the series is $-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \cdots$

This series does not converge, because the limit of the individual terms is not zero.

When $x = -1$, the series is $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots$

This series does not converge, because the limit of the individual terms is not zero.

Thus, the interval of convergence is $-1 < x < 1$.

5 :
- 1 : sets up ratio
- 1 : computes limit of ratio
- 1 : identifies radius of convergence
- 1 : considers both endpoints
- 1 : analysis/conclusion for both endpoints

(b) $f'(x) = -\frac{1}{2} + \frac{4}{3}x - \frac{9}{4}x^2 + \cdots$ and $f'(0) = -\frac{1}{2}$.

$g'(x) = -\frac{1}{2!} + \frac{2}{4!}x - \frac{3}{6!}x^2 + \cdots$ and $g'(0) = -\frac{1}{2}$.

$y'(0) = f'(0) - g'(0) = 0$

$f''(0) = \frac{4}{3}$ and $g''(0) = \frac{2}{4!} = \frac{1}{12}$.

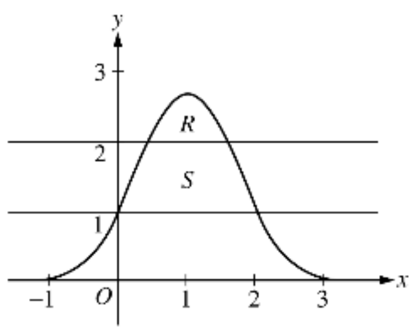
Thus, $y''(0) = \frac{4}{3} - \frac{1}{12} > 0$.

Since $y'(0) = 0$ and $y''(0) > 0$, y has a relative minimum at $x = 0$.

4 :
- 1 : $y'(0)$
- 1 : $y''(0)$
- 1 : conclusion
- 1 : reasoning

CALCULUS BC
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.



1. Let R be the region bounded by the graph of $y = e^{2x-x^2}$ and the horizontal line $y = 2$, and let S be the region bounded by the graph of $y = e^{2x-x^2}$ and the horizontal lines $y = 1$ and $y = 2$, as shown above.

   (a) Find the area of R.

   (b) Find the area of S.

   (c) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 1$.

---

2. An object moving along a curve in the xy-plane is at position $(x(t), y(t))$ at time t with

$$\frac{dx}{dt} = \arctan\left(\frac{t}{1+t}\right) \text{ and } \frac{dy}{dt} = \ln\left(t^2 + 1\right)$$

   for $t \geq 0$. At time $t = 0$, the object is at position $(-3, -4)$. (Note: $\tan^{-1} x = \arctan x$)

   (a) Find the speed of the object at time $t = 4$.

   (b) Find the total distance traveled by the object over the time interval $0 \leq t \leq 4$.

   (c) Find $x(4)$.

   (d) For $t > 0$, there is a point on the curve where the line tangent to the curve has slope 2. At what time t is the object at this point? Find the acceleration vector at this point.

---

3. The wind chill is the temperature, in degrees Fahrenheit (°F), a human feels based on the air temperature, in degrees Fahrenheit, and the wind velocity v, in miles per hour (mph). If the air temperature is 32°F, then the wind chill is given by $W(v) = 55.6 - 22.1v^{0.16}$ and is valid for $5 \le v \le 60$.

(a) Find $W'(20)$. Using correct units, explain the meaning of $W'(20)$ in terms of the wind chill.

(b) Find the average rate of change of W over the interval $5 \le v \le 60$. Find the value of v at which the instantaneous rate of change of W is equal to the average rate of change of W over the interval $5 \le v \le 60$.

(c) Over the time interval $0 \le t \le 4$ hours, the air temperature is a constant 32°F. At time $t = 0$, the wind velocity is $v = 20$ mph. If the wind velocity increases at a constant rate of 5 mph per hour, what is the rate of change of the wind chill with respect to time at $t = 3$ hours? Indicate units of measure.

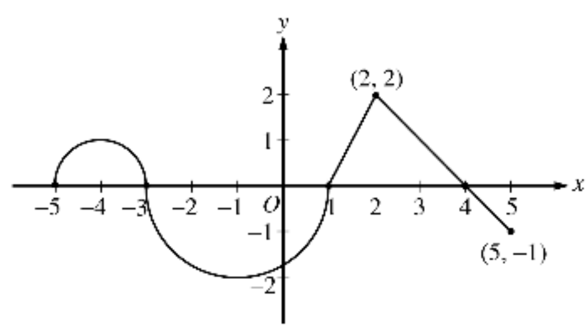
CALCULUS BC

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.



Graph of f'

4. Let f be a function defined on the closed interval $-5 \le x \le 5$ with $f(1) = 3$. The graph of f', the derivative of f, consists of two semicircles and two line segments, as shown above.

(a) For $-5 < x < 5$, find all values x at which f has a relative maximum. Justify your answer.

(b) For $-5 < x < 5$, find all values x at which the graph of f has a point of inflection. Justify your answer.

(c) Find all intervals on which the graph of f is concave up and also has positive slope. Explain your reasoning.

(d) Find the absolute minimum value of $f(x)$ over the closed interval $-5 \le x \le 5$. Explain your reasoning.

---

5. Consider the differential equation $\frac{dy}{dx} = 3x + 2y + 1$.

(a) Find $\frac{d^2y}{dx^2}$ in terms of x and y.

(b) Find the values of the constants m, b, and r for which $y = mx + b + e^{rx}$ is a solution to the differential equation.

(c) Let $y = f(x)$ be a particular solution to the differential equation with the initial condition $f(0) = -2$. Use Euler's method, starting at $x = 0$ with a step size of $\frac{1}{2}$, to approximate $f(1)$. Show the work that leads to your answer.

(d) Let $y = g(x)$ be another solution to the differential equation with the initial condition $g(0) = k$, where k is a constant. Euler's method, starting at $x = 0$ with a step size of 1, gives the approximation $g(1) \approx 0$. Find the value of k.

---

6. Let f be the function given by $f(x) = 6e^{-x/3}$ for all x.

(a) Find the first four nonzero terms and the general term for the Taylor series for f about $x = 0$.

(b) Let g be the function given by $g(x) = \int_0^x f(t)\,dt$. Find the first four nonzero terms and the general term for the Taylor series for g about $x = 0$.

(c) The function h satisfies $h(x) = k\,f'(ax)$ for all x, where a and k are constants. The Taylor series for h about $x = 0$ is given by

$$h(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots.$$

Find the values of a and k.

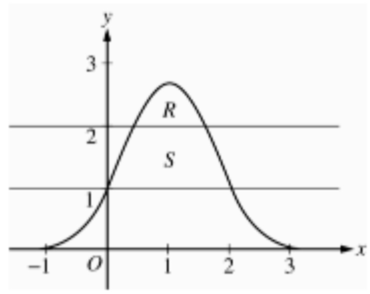
Question 1

Let R be the region bounded by the graph of $y = e^{2x-x^2}$ and the horizontal line $y = 2$, and let S be the region bounded by the graph of $y = e^{2x-x^2}$ and the horizontal lines $y = 1$ and $y = 2$, as shown above.

(a) Find the area of R.

(b) Find the area of S.

(c) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 1$.



$e^{2x-x^2} = 2$ when $x = 0.446057, 1.553943$

Let $P = 0.446057$ and $Q = 1.553943$

(a) Area of $R = \int_P^Q \left(e^{2x-x^2} - 2\right) dx = 0.514$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits} \\ 1 : \text{answer} \end{cases}$

(b) $e^{2x-x^2} = 1$ when $x = 0, 2$

Area of $S = \int_0^2 \left(e^{2x-x^2} - 1\right) dx -$ Area of R

$= 2.06016 -$ Area of $R = 1.546$

OR

$\int_0^P \left(e^{2x-x^2} - 1\right) dx + (Q - P) \cdot 1 + \int_Q^2 \left(e^{2x-x^2} - 1\right) dx$

$= 0.219064 + 1.107886 + 0.219064 = 1.546$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits} \\ 1 : \text{answer} \end{cases}$

(c) Volume $= \pi \int_P^Q \left(\left(e^{2x-x^2} - 1\right)^2 - (2-1)^2 \right) dx$

$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{constant and limits} \end{cases}$

Question 2

An object moving along a curve in the xy-plane is at position $(x(t), y(t))$ at time t with

$$\frac{dx}{dt} = \arctan\left(\frac{t}{1+t}\right) \text{ and } \frac{dy}{dt} = \ln\left(t^2 + 1\right)$$

for $t \geq 0$. At time $t = 0$, the object is at position $(-3, -4)$. (Note: $\tan^{-1} x = \arctan x$)

(a) Find the speed of the object at time $t = 4$.

(b) Find the total distance traveled by the object over the time interval $0 \leq t \leq 4$.

(c) Find $x(4)$.

(d) For $t > 0$, there is a point on the curve where the line tangent to the curve has slope 2. At what time t is the object at this point? Find the acceleration vector at this point.

(a) Speed $= \sqrt{x'(4)^2 + y'(4)^2} = 2.912$	$1:$ speed at $t = 4$
(b) Distance $= \int_0^4 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = 6.423$	$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$
(c) $x(4) = x(0) + \int_0^4 x'(t)\,dt$ $= -3 + 2.10794 = -0.892$	$3: \begin{cases} 2: \begin{cases} 1: \text{integrand} \\ 1: \text{uses } x(0) = -3 \end{cases} \\ 1: \text{answer} \end{cases}$
(d) The slope is 2, so $\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = 2$, or $\ln\left(t^2+1\right) = 2\arctan\left(\frac{t}{1+t}\right)$. Since $t > 0$, $t = 1.35766$. At this time, the acceleration is $\langle x''(t), y''(t)\rangle\big\|_{t=1.35766} = \langle 0.135, 0.955\rangle$.	$3: \begin{cases} 1: \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = 2 \\ 1: t\text{-value} \\ 1: \text{values for } x'' \text{ and } y'' \end{cases}$

Question 3

The wind chill is the temperature, in degrees Fahrenheit ($^\circ$F), a human feels based on the air temperature, in degrees Fahrenheit, and the wind velocity v, in miles per hour (mph). If the air temperature is 32°F, then the wind chill is given by $W(v) = 55.6 - 22.1v^{0.16}$ and is valid for $5 \le v \le 60$.

(a) Find $W'(20)$. Using correct units, explain the meaning of $W'(20)$ in terms of the wind chill.

(b) Find the average rate of change of W over the interval $5 \le v \le 60$. Find the value of v at which the instantaneous rate of change of W is equal to the average rate of change of W over the interval $5 \le v \le 60$.

(c) Over the time interval $0 \le t \le 4$ hours, the air temperature is a constant 32°F. At time $t = 0$, the wind velocity is $v = 20$ mph. If the wind velocity increases at a constant rate of 5 mph per hour, what is the rate of change of the wind chill with respect to time at $t = 3$ hours? Indicate units of measure.

(a) $W'(20) = -22.1 \cdot 0.16 \cdot 20^{-0.84} = -0.285$ or -0.286

When $v = 20$ mph, the wind chill is decreasing at $0.286\ ^\circ\text{F/mph}$.

$2: \begin{cases} 1: \text{value} \\ 1: \text{explanation} \end{cases}$

(b) The average rate of change of W over the interval $5 \le v \le 60$ is $\dfrac{W(60) - W(5)}{60 - 5} = -0.253$ or -0.254.

$W'(v) = \dfrac{W(60) - W(5)}{60 - 5}$ when $v = 23.011$.

$3: \begin{cases} 1: \text{average rate of change} \\ 1: W'(v) = \text{average rate of change} \\ 1: \text{value of } v \end{cases}$

(c) $\left.\dfrac{dW}{dt}\right|_{t=3} = \left.\left(\dfrac{dW}{dv} \cdot \dfrac{dv}{dt}\right)\right|_{t=3} = W'(35) \cdot 5 = -0.892\ ^\circ\text{F/hr}$

OR

$W = 55.6 - 22.1(20 + 5t)^{0.16}$

$\left.\dfrac{dW}{dt}\right|_{t=3} = -0.892\ ^\circ\text{F/hr}$

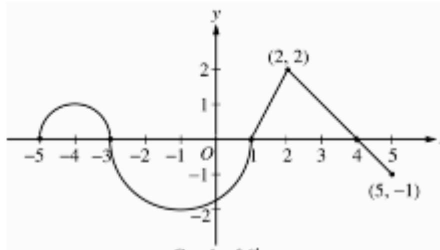
$3: \begin{cases} 1: \dfrac{dv}{dt} = 5 \\ 1: \text{uses } v(3) = 35, \\ \quad \text{or} \\ \quad \text{uses } v(t) = 20 + 5t \\ 1: \text{answer} \end{cases}$

Units of $^\circ$F/mph in (a) and $^\circ$F/hr in (c)

1 : units in (a) and (c)

Question 4

Let f be a function defined on the closed interval $-5 \le x \le 5$ with $f(1) = 3$. The graph of f', the derivative of f, consists of two semicircles and two line segments, as shown above.



Graph of f'

(a) For $-5 < x < 5$, find all values x at which f has a relative maximum. Justify your answer.

(b) For $-5 < x < 5$, find all values x at which the graph of f has a point of inflection. Justify your answer.

(c) Find all intervals on which the graph of f is concave up and also has positive slope. Explain your reasoning.

(d) Find the absolute minimum value of $f(x)$ over the closed interval $-5 \le x \le 5$. Explain your reasoning.

(a) $f'(x) = 0$ at $x = -3, 1, 4$
f' changes from positive to negative at -3 and 4.
Thus, f has a relative maximum at $x = -3$ and at $x = 4$.

$2: \begin{cases} 1: x\text{-values} \\ 1: \text{justification} \end{cases}$

(b) f' changes from increasing to decreasing, or vice versa, at $x = -4, -1$, and 2. Thus, the graph of f has points of inflection when $x = -4, -1$, and 2.

$2: \begin{cases} 1: x\text{-values} \\ 1: \text{justification} \end{cases}$

(c) The graph of f is concave up with positive slope where f' is increasing and positive: $-5 < x < -4$ and $1 < x < 2$.

$2: \begin{cases} 1: \text{intervals} \\ 1: \text{explanation} \end{cases}$

(d) Candidates for the absolute minimum are where f' changes from negative to positive (at $x = 1$) and at the endpoints ($x = -5, 5$).

$f(-5) = 3 + \int_1^{-5} f'(x)\,dx = 3 - \frac{\pi}{2} + 2\pi > 3$

$f(1) = 3$

$f(5) = 3 + \int_1^{5} f'(x)\,dx = 3 + \frac{3 \cdot 2}{2} - \frac{1}{2} > 3$

The absolute minimum value of f on $[-5, 5]$ is $f(1) = 3$.

$3: \begin{cases} 1: \text{identifies } x = 1 \text{ as a candidate} \\ 1: \text{considers endpoints} \\ 1: \text{value and explanation} \end{cases}$

Question 5

Consider the differential equation $\frac{dy}{dx} = 3x + 2y + 1$.

(a) Find $\frac{d^2y}{dx^2}$ in terms of x and y.

(b) Find the values of the constants m, b, and r for which $y = mx + b + e^{rx}$ is a solution to the differential equation.

(c) Let $y = f(x)$ be a particular solution to the differential equation with the initial condition $f(0) = -2$. Use Euler's method, starting at $x = 0$ with a step size of $\frac{1}{2}$, to approximate $f(1)$. Show the work that leads to your answer.

(d) Let $y = g(x)$ be another solution to the differential equation with the initial condition $g(0) = k$, where k is a constant. Euler's method, starting at $x = 0$ with a step size of 1, gives the approximation $g(1) \approx 0$. Find the value of k.

(a) $\frac{d^2y}{dx^2} = 3 + 2\frac{dy}{dx} = 3 + 2(3x + 2y + 1) = 6x + 4y + 5$	$2: \begin{cases} 1: 3 + 2\frac{dy}{dx} \\ 1: \text{answer} \end{cases}$
(b) If $y = mx + b + e^{rx}$ is a solution, then $m + re^{rx} = 3x + 2(mx + b + e^{rx}) + 1$. If $r \neq 0$: $m = 2b + 1$, $r = 2$, $0 = 3 + 2m$, so $m = -\frac{3}{2}$, $r = 2$, and $b = -\frac{5}{4}$. OR If $r = 0$: $m = 2b + 3$, $r = 0$, $0 = 3 + 2m$, so $m = -\frac{3}{2}$, $r = 0$, $b = -\frac{9}{4}$.	$3: \begin{cases} 1: \frac{dy}{dx} = m + re^{rx} \\ 1: \text{value for } r \\ 1: \text{values for } m \text{ and } b \end{cases}$
(c) $f\left(\frac{1}{2}\right) \approx f(0) + f'(0)\cdot\frac{1}{2} = -2 + (-3)\cdot\frac{1}{2} = -\frac{7}{2}$ $f'\left(\frac{1}{2}\right) \approx 3\left(\frac{1}{2}\right) + 2\left(-\frac{7}{2}\right) + 1 = -\frac{9}{2}$ $f(1) \approx f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right)\cdot\frac{1}{2} = -\frac{7}{2} + \left(-\frac{9}{2}\right)\cdot\frac{1}{2} = -\frac{23}{4}$	$2: \begin{cases} 1: \text{Euler's method with 2 steps} \\ 1: \text{Euler's approximation for } f(1) \end{cases}$
(d) $g'(0) = 3\cdot 0 + 2\cdot k + 1 = 2k + 1$ $g(1) \approx g(0) + g'(0)\cdot 1 = k + (2k + 1) = 3k + 1 = 0$ $k = -\frac{1}{3}$	$2: \begin{cases} 1: g(0) + g'(0)\cdot 1 \\ 1: \text{value of } k \end{cases}$

Question 6

Let f be the function given by $f(x) = 6e^{-x/3}$ for all x.

(a) Find the first four nonzero terms and the general term for the Taylor series for f about $x = 0$.

(b) Let g be the function given by $g(x) = \int_0^x f(t)\,dt$. Find the first four nonzero terms and the general term for the Taylor series for g about $x = 0$.

(c) The function h satisfies $h(x) = kf'(ax)$ for all x, where a and k are constants. The Taylor series for h about $x = 0$ is given by

$$h(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots.$$

Find the values of a and k.

(a) $$f(x) = 6\left[1 - \frac{x}{3} + \frac{x^2}{2!3^2} - \frac{x^3}{3!3^3} + \cdots + \frac{(-1)^n x^n}{n!3^n} + \cdots\right]$$
$$= 6 - 2x + \frac{x^2}{3} - \frac{x^3}{27} + \cdots + \frac{6(-1)^n x^n}{n!3^n} + \cdots$$

3 : $\begin{cases} 1 : \text{two of } 6, -2x, \frac{x^2}{3}, -\frac{x^3}{27} \\ 1 : \text{remaining terms} \\ 1 : \text{general term} \end{cases}$

$\langle -1 \rangle$ missing factor of 6

(b) $g(0) = 0$ and $g'(x) = f(x)$, so
$$g(x) = 6\left[x - \frac{x^2}{6} + \frac{x^3}{3!3^2} - \frac{x^4}{4!3^3} + \cdots + \frac{(-1)^n x^{n+1}}{(n+1)!3^n} + \cdots\right]$$
$$= 6x - x^2 + \frac{x^3}{9} - \frac{x^4}{4(27)} + \cdots + \frac{6(-1)^n x^{n+1}}{(n+1)!3^n} + \cdots$$

3 : $\begin{cases} 1 : \text{two terms} \\ 1 : \text{remaining terms} \\ 1 : \text{general term} \end{cases}$

$\langle -1 \rangle$ missing factor of 6

(c) $f'(x) = -2e^{-x/3}$, so $h(x) = -2ke^{-ax/3}$

$h(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots = e^x$

$-2ke^{-ax/3} = e^x$

$\frac{-a}{3} = 1$ and $-2k = 1$

$a = -3$ and $k = -\frac{1}{2}$

OR

$f'(x) = -2 + \frac{2}{3}x + \cdots$, so

$h(x) = kf'(ax) = -2k + \frac{2}{3}akx + \cdots$

$h(x) = 1 + x + \cdots$

$-2k = 1$ and $\frac{2}{3}ak = 1$

$k = -\frac{1}{2}$ and $a = -3$

3 : $\begin{cases} 1 : \text{computes } kf'(ax) \\ 1 : \text{recognizes } h(x) = e^x, \\ \quad\quad \text{or} \\ \quad \text{equates 2 series for } h(x) \\ 1 : \text{values for } a \text{ and } k \end{cases}$

CALCULUS BC
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.

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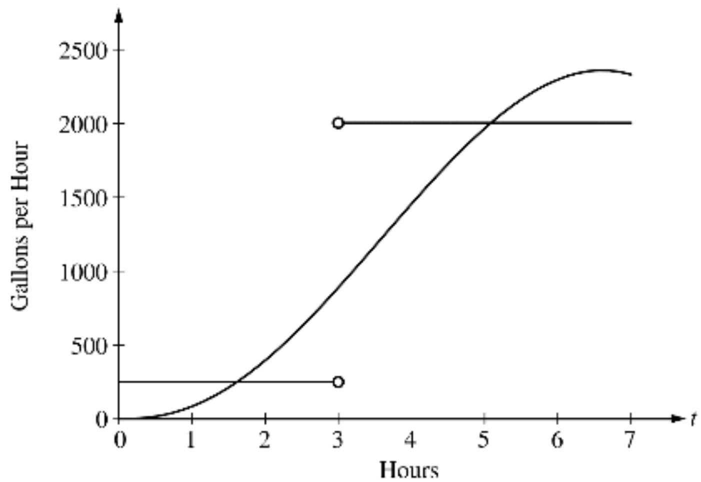
1. Let R be the region in the first and second quadrants bounded above by the graph of $y = \dfrac{20}{1 + x^2}$ and below by the horizontal line $y = 2$.

   (a) Find the area of R.

   (b) Find the volume of the solid generated when R is rotated about the x-axis.

   (c) The region R is the base of a solid. For this solid, the cross sections perpendicular to the x-axis are semicircles. Find the volume of this solid.



2. The amount of water in a storage tank, in gallons, is modeled by a continuous function on the time interval $0 \le t \le 7$, where t is measured in hours. In this model, rates are given as follows:

   (i) The rate at which water enters the tank is $f(t) = 100t^2 \sin\left(\sqrt{t}\right)$ gallons per hour for $0 \le t \le 7$.

   (ii) The rate at which water leaves the tank is

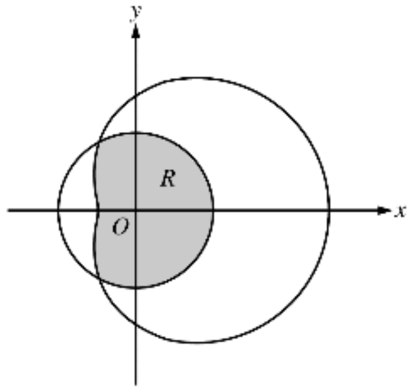
$$g(t) = \begin{cases} 250 & \text{for } 0 \le t < 3 \\ 2000 & \text{for } 3 < t \le 7 \end{cases} \text{ gallons per hour.}$$

The graphs of f and g, which intersect at $t = 1.617$ and $t = 5.076$, are shown in the figure above. At time $t = 0$, the amount of water in the tank is 5000 gallons.

(a) How many gallons of water enter the tank during the time interval $0 \le t \le 7$? Round your answer to the nearest gallon.

(b) For $0 \le t \le 7$, find the time intervals during which the amount of water in the tank is decreasing. Give a reason for each answer.

(c) For $0 \le t \le 7$, at what time t is the amount of water in the tank greatest? To the nearest gallon, compute the amount of water at this time. Justify your answer.



3. The graphs of the polar curves $r = 2$ and $r = 3 + 2\cos\theta$ are shown in the figure above. The curves intersect when $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$.

   (a) Let R be the region that is inside the graph of $r = 2$ and also inside the graph of $r = 3 + 2\cos\theta$, as shaded in the figure above. Find the area of R.

   (b) A particle moving with nonzero velocity along the polar curve given by $r = 3 + 2\cos\theta$ has position $(x(t), y(t))$ at time t, with $\theta = 0$ when $t = 0$. This particle moves along the curve so that $\frac{dr}{dt} = \frac{dr}{d\theta}$. Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{3}$ and interpret your answer in terms of the motion of the particle.

   (c) For the particle described in part (b), $\frac{dy}{dt} = \frac{dy}{d\theta}$. Find the value of $\frac{dy}{dt}$ at $\theta = \frac{\pi}{3}$ and interpret your answer in terms of the motion of the particle.

CALCULUS BC

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.

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4. Let f be the function defined for $x > 0$, with $f(e) = 2$ and f', the first derivative of f, given by $f'(x) = x^2 \ln x$.

   (a) Write an equation for the line tangent to the graph of f at the point $(e, 2)$.

   (b) Is the graph of f concave up or concave down on the interval $1 < x < 3$? Give a reason for your answer.

   (c) Use antidifferentiation to find $f(x)$.

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t (minutes)	0	2	5	7	11	12
$r'(t)$ (feet per minute)	5.7	4.0	2.0	1.2	0.6	0.5

5. The volume of a spherical hot air balloon expands as the air inside the balloon is heated. The radius of the balloon, in feet, is modeled by a twice-differentiable function r of time t, where t is measured in minutes. For $0 < t < 12$, the graph of r is concave down. The table above gives selected values of the rate of change, $r'(t)$, of the radius of the balloon over the time interval $0 \le t \le 12$. The radius of the balloon is 30 feet when $t = 5$. (Note: The volume of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$.)

   (a) Estimate the radius of the balloon when $t = 5.4$ using the tangent line approximation at $t = 5$. Is your estimate greater than or less than the true value? Give a reason for your answer.

   (b) Find the rate of change of the volume of the balloon with respect to time when $t = 5$. Indicate units of measure.

   (c) Use a right Riemann sum with the five subintervals indicated by the data in the table to approximate $\int_0^{12} r'(t)\,dt$. Using correct units, explain the meaning of $\int_0^{12} r'(t)\,dt$ in terms of the radius of the balloon.

   (d) Is your approximation in part (c) greater than or less than $\int_0^{12} r'(t)\,dt$? Give a reason for your answer.

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6. Let f be the function given by $f(x) = e^{-x^2}$.

   (a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 0$.

   (b) Use your answer to part (a) to find $\lim_{x\to 0} \frac{1 - x^2 - f(x)}{x^4}$.

   (c) Write the first four nonzero terms of the Taylor series for $\int_0^x e^{-t^2}\,dt$ about $x = 0$. Use the first two terms of your answer to estimate $\int_0^{1/2} e^{-t^2}\,dt$.

   (d) Explain why the estimate found in part (c) differs from the actual value of $\int_0^{1/2} e^{-t^2}\,dt$ by less than $\frac{1}{200}$.

Question 1

Let R be the region in the first and second quadrants bounded above by the graph of $y = \dfrac{20}{1+x^2}$ and below by the horizontal line $y = 2$.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is rotated about the x-axis.

(c) The region R is the base of a solid. For this solid, the cross sections perpendicular to the x-axis are semicircles. Find the volume of this solid.

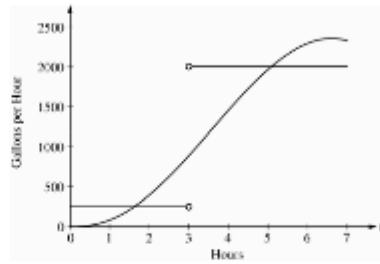
$\dfrac{20}{1+x^2} = 2$ when $x = \pm 3$	1 : correct limits in an integral in (a), (b), or (c)
(a) Area $= \displaystyle\int_{-3}^{3} \left(\frac{20}{1+x^2} - 2 \right) dx = 37.961$ or 37.962	$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$
(b) Volume $= \pi \displaystyle\int_{-3}^{3} \left(\left(\frac{20}{1+x^2} \right)^2 - 2^2 \right) dx = 1871.190$	$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{answer} \end{cases}$
(c) Volume $= \dfrac{\pi}{2} \displaystyle\int_{-3}^{3} \left(\frac{1}{2} \left(\frac{20}{1+x^2} - 2 \right) \right)^2 dx$ $= \dfrac{\pi}{8} \displaystyle\int_{-3}^{3} \left(\frac{20}{1+x^2} - 2 \right)^2 dx = 174.268$	$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

Question 2

The amount of water in a storage tank, in gallons, is modeled by a continuous function on the time interval $0 \le t \le 7$, where t is measured in hours. In this model, rates are given as follows:

(i) The rate at which water enters the tank is $f(t) = 100t^2 \sin(\sqrt{t})$ gallons per hour for $0 \le t \le 7$.

(ii) The rate at which water leaves the tank is $g(t) = \begin{cases} 250 & \text{for } 0 \le t < 3 \\ 2000 & \text{for } 3 < t \le 7 \end{cases}$ gallons per hour.



The graphs of f and g, which intersect at $t = 1.617$ and $t = 5.076$, are shown in the figure above. At time $t = 0$, the amount of water in the tank is 5000 gallons.

(a) How many gallons of water enter the tank during the time interval $0 \le t \le 7$? Round your answer to the nearest gallon.

(b) For $0 \le t \le 7$, find the time intervals during which the amount of water in the tank is decreasing. Give a reason for each answer.

(c) For $0 \le t \le 7$, at what time t is the amount of water in the tank greatest? To the nearest gallon, compute the amount of water at this time. Justify your answer.

(a) $\int_0^7 f(t)\,dt \approx 8264$ gallons

$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$

(b) The amount of water in the tank is decreasing on the intervals $0 \le t \le 1.617$ and $3 \le t \le 5.076$ because $f(t) < g(t)$ for $0 \le t < 1.617$ and $3 < t < 5.076$.

$2: \begin{cases} 1: \text{intervals} \\ 1: \text{reason} \end{cases}$

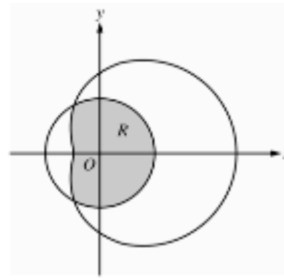
(c) Since $f(t) - g(t)$ changes sign from positive to negative only at $t = 3$, the candidates for the absolute maximum are at $t = 0$, 3, and 7.

t (hours)	gallons of water
0	5000
3	$5000 + \int_0^3 f(t)\,dt - 250(3) = 5126.591$
7	$5126.591 + \int_3^7 f(t)\,dt - 2000(4) = 4513.807$

The amount of water in the tank is greatest at 3 hours. At that time, the amount of water in the tank, rounded to the nearest gallon, is 5127 gallons.

$5: \begin{cases} 1: \text{identifies } t = 3 \text{ as a candidate} \\ 1: \text{integrand} \\ 1: \text{amount of water at } t = 3 \\ 1: \text{amount of water at } t = 7 \\ 1: \text{conclusion} \end{cases}$

Question 3



The graphs of the polar curves $r = 2$ and $r = 3 + 2\cos\theta$ are shown in the figure above. The curves intersect when $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$.

(a) Let R be the region that is inside the graph of $r = 2$ and also inside the graph of $r = 3 + 2\cos\theta$, as shaded in the figure above. Find the area of R.

(b) A particle moving with nonzero velocity along the polar curve given by $r = 3 + 2\cos\theta$ has position $(x(t), y(t))$ at time t, with $\theta = 0$ when $t = 0$. This particle moves along the curve so that $\frac{dr}{dt} = \frac{dr}{d\theta}$. Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{3}$ and interpret your answer in terms of the motion of the particle.

(c) For the particle described in part (b), $\frac{dy}{dt} = \frac{dy}{d\theta}$. Find the value of $\frac{dy}{dt}$ at $\theta = \frac{\pi}{3}$ and interpret your answer in terms of the motion of the particle.

(a) Area $= \frac{2}{3}\pi(2)^2 + \frac{1}{2}\int_{2\pi/3}^{4\pi/3}(3 + 2\cos\theta)^2\, d\theta$

$= 10.370$

4 :
- 1 : area of circular sector
- 2 : integral for section of limaçon
  - 1 : integrand
  - 1 : limits and constant
- 1 : answer

(b) $\left.\frac{dr}{dt}\right|_{\theta=\pi/3} = \left.\frac{dr}{d\theta}\right|_{\theta=\pi/3} = -1.732$

The particle is moving closer to the origin, since $\frac{dr}{dt} < 0$ and $r > 0$ when $\theta = \frac{\pi}{3}$.

2 :
- 1 : $\left.\frac{dr}{dt}\right|_{\theta=\pi/3}$
- 1 : interpretation

(c) $y = r\sin\theta = (3 + 2\cos\theta)\sin\theta$

$\left.\frac{dy}{dt}\right|_{\theta=\pi/3} = \left.\frac{dy}{d\theta}\right|_{\theta=\pi/3} = 0.5$

The particle is moving away from the x-axis, since $\frac{dy}{dt} > 0$ and $y > 0$ when $\theta = \frac{\pi}{3}$.

3 :
- 1 : expression for y in terms of θ
- 1 : $\left.\frac{dy}{dt}\right|_{\theta=\pi/3}$
- 1 : interpretation

Question 4

Let f be the function defined for $x > 0$, with $f(e) = 2$ and f', the first derivative of f, given by $f'(x) = x^2 \ln x$.

(a) Write an equation for the line tangent to the graph of f at the point $(e, 2)$.

(b) Is the graph of f concave up or concave down on the interval $1 < x < 3$? Give a reason for your answer.

(c) Use antidifferentiation to find $f(x)$.

(a) $f'(e) = e^2$

An equation for the line tangent to the graph of f at the point $(e, 2)$ is $y - 2 = e^2(x - e)$.

$2 : \begin{cases} 1 : f'(e) \\ 1 : \text{equation of tangent line} \end{cases}$

(b) $f''(x) = x + 2x \ln x$.

For $1 < x < 3$, $x > 0$ and $\ln x > 0$, so $f''(x) > 0$. Thus, the graph of f is concave up on $(1, 3)$.

$3 : \begin{cases} 2 : f''(x) \\ 1 : \text{answer with reason} \end{cases}$

(c) Since $f(x) = \int (x^2 \ln x)\, dx$, we consider integration by parts.

$$u = \ln x \qquad dv = x^2\, dx$$
$$du = \frac{1}{x}\, dx \qquad v = \int (x^2)\, dx = \frac{1}{3}x^3$$

Therefore,

$$f(x) = \int (x^2 \ln x)\, dx$$
$$= \frac{1}{3}x^3 \ln x - \int \left(\frac{1}{3}x^3 \cdot \frac{1}{x}\right) dx$$
$$= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C.$$

Since $f(e) = 2$, $2 = \frac{e^3}{3} - \frac{e^3}{9} + C$ and $C = 2 - \frac{2}{9}e^3$.

Thus, $f(x) = \frac{x^3}{3}\ln x - \frac{1}{9}x^3 + 2 - \frac{2}{9}e^3$.

$4 : \begin{cases} 2 : \text{antiderivative} \\ 1 : \text{uses } f(e) = 2 \\ 1 : \text{answer} \end{cases}$

Question 5

t (minutes)	0	2	5	7	11	12
$r'(t)$ (feet per minute)	5.7	4.0	2.0	1.2	0.6	0.5

The volume of a spherical hot air balloon expands as the air inside the balloon is heated. The radius of the balloon, in feet, is modeled by a twice-differentiable function r of time t, where t is measured in minutes. For $0 < t < 12$, the graph of r is concave down. The table above gives selected values of the rate of change, $r'(t)$, of the radius of the balloon over the time interval $0 \le t \le 12$. The radius of the balloon is 30 feet when $t = 5$. (Note: The volume of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$.)

(a) Estimate the radius of the balloon when $t = 5.4$ using the tangent line approximation at $t = 5$. Is your estimate greater than or less than the true value? Give a reason for your answer.

(b) Find the rate of change of the volume of the balloon with respect to time when $t = 5$. Indicate units of measure.

(c) Use a right Riemann sum with the five subintervals indicated by the data in the table to approximate $\int_0^{12} r'(t)\,dt$. Using correct units, explain the meaning of $\int_0^{12} r'(t)\,dt$ in terms of the radius of the balloon.

(d) Is your approximation in part (c) greater than or less than $\int_0^{12} r'(t)\,dt$? Give a reason for your answer.

(a) $r(5.4) \approx r(5) + r'(5)\Delta t = 30 + 2(0.4) = 30.8$ ft Since the graph of r is concave down on the interval $5 < t < 5.4$, this estimate is greater than $r(5.4)$.	$2: \begin{cases} 1: \text{estimate} \\ 1: \text{conclusion with reason} \end{cases}$
(b) $\frac{dV}{dt} = 3\left(\frac{4}{3}\right)\pi r^2 \frac{dr}{dt}$ $\left.\frac{dV}{dt}\right\rvert_{t=5} = 4\pi(30)^2 2 = 7200\pi \text{ ft}^3/\text{min}$	$3: \begin{cases} 2: \frac{dV}{dt} \\ 1: \text{answer} \end{cases}$
(c) $\int_0^{12} r'(t)\,dt \approx 2(4.0) + 3(2.0) + 2(1.2) + 4(0.6) + 1(0.5)$ $= 19.3$ ft $\int_0^{12} r'(t)\,dt$ is the change in the radius, in feet, from $t = 0$ to $t = 12$ minutes.	$2: \begin{cases} 1: \text{approximation} \\ 1: \text{explanation} \end{cases}$
(d) Since r is concave down, r' is decreasing on $0 < t < 12$. Therefore, this approximation, 19.3 ft, is less than $\int_0^{12} r'(t)\,dt$.	1 : conclusion with reason
Units of ft^3/min in part (b) and ft in part (c)	1 : units in (b) and (c)

Question 6

Let f be the function given by $f(x) = e^{-x^2}$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 0$.

(b) Use your answer to part (a) to find $\lim_{x\to 0}\frac{1 - x^2 - f(x)}{x^4}$.

(c) Write the first four nonzero terms of the Taylor series for $\int_0^x e^{-t^2}\,dt$ about $x = 0$. Use the first two terms of your answer to estimate $\int_0^{1/2} e^{-t^2}\,dt$.

(d) Explain why the estimate found in part (c) differs from the actual value of $\int_0^{1/2} e^{-t^2}\,dt$ by less than $\frac{1}{200}$.

(a) $e^{-x^2} = 1 + \frac{(-x^2)}{1!} + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \cdots + \frac{(-x^2)^n}{n!} + \cdots$ $= 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots + \frac{(-1)^n x^{2n}}{n!} + \cdots$	$3: \begin{cases} 1: \text{two of } 1, -x^2, \frac{x^4}{2}, -\frac{x^6}{6} \\ 1: \text{remaining terms} \\ 1: \text{general term} \end{cases}$
(b) $\frac{1 - x^2 - f(x)}{x^4} = -\frac{1}{2} + \frac{x^2}{6} + \sum_{n=4}^{\infty} \frac{(-1)^{n+1} x^{2n-4}}{n!}$ Thus, $\lim_{x\to 0}\left(\frac{1 - x^2 - f(x)}{x^4}\right) = -\frac{1}{2}$.	1 : answer
(c) $\int_0^x e^{-t^2}\,dt = \int_0^x \left(1 - t^2 + \frac{t^4}{2} - \frac{t^6}{6} + \cdots + \frac{(-1)^n t^{2n}}{n!} + \cdots\right) dt$ $= x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \cdots$ Using the first two terms of this series, we estimate that $\int_0^{1/2} e^{-t^2}\,dt \approx \frac{1}{2} - \left(\frac{1}{3}\right)\left(\frac{1}{8}\right) = \frac{11}{24}$.	$3: \begin{cases} 1: \text{two terms} \\ 1: \text{remaining terms} \\ 1: \text{estimate} \end{cases}$
(d) $\left\lvert \int_0^{1/2} e^{-t^2}\,dt - \frac{11}{24} \right\rvert < \left(\frac{1}{2}\right)^5 \cdot \frac{1}{10} = \frac{1}{320} < \frac{1}{200}$, since $\int_0^{1/2} e^{-t^2}\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\right)^{2n+1}}{n!(2n+1)}$, which is an alternating series with individual terms that decrease in absolute value to 0.	$2: \begin{cases} 1: \text{uses the third term as the error bound} \\ 1: \text{explanation} \end{cases}$

CALCULUS BC
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.

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1. A particle moving along a curve in the xy-plane has position $(x(t), y(t))$ at time $t \geq 0$ with

$$\frac{dx}{dt} = \sqrt{3t} \text{ and } \frac{dy}{dt} = 3\cos\left(\frac{t^2}{2}\right).$$

The particle is at position $(1, 5)$ at time $t = 4$.

(a) Find the acceleration vector at time $t = 4$.

(b) Find the y-coordinate of the position of the particle at time $t = 0$.

(c) On the interval $0 \leq t \leq 4$, at what time does the speed of the particle first reach 3.5 ?

(d) Find the total distance traveled by the particle over the time interval $0 \leq t \leq 4$.

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2. For time $t \geq 0$ hours, let $r(t) = 120\left(1 - e^{-10t^2}\right)$ represent the speed, in kilometers per hour, at which a car travels along a straight road. The number of liters of gasoline used by the car to travel x kilometers is modeled by $g(x) = 0.05x\left(1 - e^{-x/2}\right)$.

(a) How many kilometers does the car travel during the first 2 hours?

(b) Find the rate of change with respect to time of the number of liters of gasoline used by the car when $t = 2$ hours. Indicate units of measure.

(c) How many liters of gasoline have been used by the car when it reaches a speed of 80 kilometers per hour?

Distance from the river's edge (feet)	0	8	14	22	24
Depth of the water (feet)	0	7	8	2	0

3. A scientist measures the depth of the Doe River at Picnic Point. The river is 24 feet wide at this location. The measurements are taken in a straight line perpendicular to the edge of the river. The data are shown in the table above. The velocity of the water at Picnic Point, in feet per minute, is modeled by $v(t) = 16 + 2\sin\left(\sqrt{t + 10}\right)$ for $0 \le t \le 120$ minutes.

   (a) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the area of the cross section of the river at Picnic Point, in square feet. Show the computations that lead to your answer.

   (b) The volumetric flow at a location along the river is the product of the cross-sectional area and the velocity of the water at that location. Use your approximation from part (a) to estimate the average value of the volumetric flow at Picnic Point, in cubic feet per minute, from $t = 0$ to $t = 120$ minutes.

   (c) The scientist proposes the function f, given by $f(x) = 8\sin\left(\frac{\pi x}{24}\right)$, as a model for the depth of the water, in feet, at Picnic Point x feet from the river's edge. Find the area of the cross section of the river at Picnic Point based on this model.

   (d) Recall that the volumetric flow is the product of the cross-sectional area and the velocity of the water at a location. To prevent flooding, water must be diverted if the average value of the volumetric flow at Picnic Point exceeds 2100 cubic feet per minute for a 20-minute period. Using your answer from part (c), find the average value of the volumetric flow during the time interval $40 \le t \le 60$ minutes. Does this value indicate that the water must be diverted?

CALCULUS BC

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.

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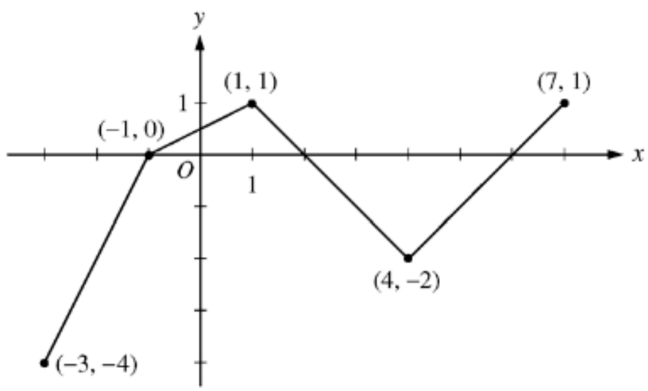
4. Let f be the function given by $f(x) = kx^2 - x^3$, where k is a positive constant. Let R be the region in the first quadrant bounded by the graph of f and the x-axis.

   (a) Find all values of the constant k for which the area of R equals 2.

   (b) For $k > 0$, write, but do not evaluate, an integral expression in terms of k for the volume of the solid generated when R is rotated about the x-axis.

   (c) For $k > 0$, write, but do not evaluate, an expression in terms of k, involving one or more integrals, that gives the perimeter of R.



Graph of g'

5. Let g be a continuous function with $g(2) = 5$. The graph of the piecewise-linear function g', the derivative of g, is shown above for $-3 \le x \le 7$.

   (a) Find the x-coordinate of all points of inflection of the graph of $y = g(x)$ for $-3 < x < 7$. Justify your answer.

   (b) Find the absolute maximum value of g on the interval $-3 \le x \le 7$. Justify your answer.

   (c) Find the average rate of change of $g(x)$ on the interval $-3 \le x \le 7$.

   (d) Find the average rate of change of $g'(x)$ on the interval $-3 \le x \le 7$. Does the Mean Value Theorem applied on the interval $-3 \le x \le 7$ guarantee a value of c, for $-3 < c < 7$, such that $g''(c)$ is equal to this average rate of change? Why or why not?

2008 AP® CALCULUS BC FREE-RESPONSE QUESTIONS (Form B)

6. Let f be the function given by $f(x) = \dfrac{2x}{1 + x^2}$.

   (a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 0$.

   (b) Does the series found in part (a), when evaluated at $x = 1$, converge to $f(1)$? Explain why or why not.

   (c) The derivative of $\ln\left(1 + x^2\right)$ is $\dfrac{2x}{1 + x^2}$. Write the first four nonzero terms of the Taylor series for $\ln\left(1 + x^2\right)$ about $x = 0$.

   (d) Use the series found in part (c) to find a rational number A such that $\left| A - \ln\left(\dfrac{5}{4}\right) \right| < \dfrac{1}{100}$. Justify your answer.

Question 1

A particle moving along a curve in the xy-plane has position $(x(t), y(t))$ at time $t \geq 0$ with

$$\frac{dx}{dt} = \sqrt{3t} \text{ and } \frac{dy}{dt} = 3\cos\left(\frac{t^2}{2}\right).$$

The particle is at position $(1, 5)$ at time $t = 4$.

(a) Find the acceleration vector at time $t = 4$.

(b) Find the y-coordinate of the position of the particle at time $t = 0$.

(c) On the interval $0 \leq t \leq 4$, at what time does the speed of the particle first reach 3.5?

(d) Find the total distance traveled by the particle over the time interval $0 \leq t \leq 4$.

(a) $a(4) = \langle x''(4), y''(4)\rangle = \langle 0.433, -11.872\rangle$	1 : answer
(b) $y(0) = 5 + \int_4^0 3\cos\left(\frac{t^2}{2}\right) dt = 1.600$ or 1.601	3 : { 1 : integrand 1 : uses $y(4) = 5$ 1 : answer
(c) Speed $= \sqrt{(x'(t))^2 + (y'(t))^2}$ $= \sqrt{3t + 9\cos^2\left(\frac{t^2}{2}\right)} = 3.5$ The particle first reaches this speed when $t = 2.225$ or 2.226.	3 : { 1 : expression for speed 1 : equation 1 : answer
(d) $\int_0^4 \sqrt{3t + 9\cos^2\left(\frac{t^2}{2}\right)}\, dt = 13.182$	2 : { 1 : integral 1 : answer

Question 2

For time $t \geq 0$ hours, let $r(t) = 120\left(1 - e^{-10t^2}\right)$ represent the speed, in kilometers per hour, at which a car travels along a straight road. The number of liters of gasoline used by the car to travel x kilometers is modeled by $g(x) = 0.05x\left(1 - e^{-x/2}\right)$.

(a) How many kilometers does the car travel during the first 2 hours?

(b) Find the rate of change with respect to time of the number of liters of gasoline used by the car when $t = 2$ hours. Indicate units of measure.

(c) How many liters of gasoline have been used by the car when it reaches a speed of 80 kilometers per hour?

(a) $\int_0^2 r(t)\,dt = 206.370$ kilometers

> $2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$

(b) $\dfrac{dg}{dt} = \dfrac{dg}{dx} \cdot \dfrac{dx}{dt}; \quad \dfrac{dx}{dt} = r(t)$

$$\left.\frac{dg}{dt}\right|_{t=2} = \left.\frac{dg}{dx}\right|_{x=206.370} \cdot r(2)$$

$$= (0.050)(120) = 6 \text{ liters / hour}$$

> $3: \begin{cases} 2: \text{uses chain rule} \\ 1: \text{answer with units} \end{cases}$

(c) Let T be the time at which the car's speed reaches 80 kilometers per hour.

Then, $r(T) = 80$ or $T = 0.331453$ hours.

At time T, the car has gone

$x(T) = \int_0^T r(t)\,dt = 10.794097$ kilometers

and has consumed $g(x(T)) = 0.537$ liters of gasoline.

> $4: \begin{cases} 1: \text{equation } r(t) = 80 \\ 2: \text{distance integral} \\ 1: \text{answer} \end{cases}$

Question 3

Distance from the river's edge (feet)	0	8	14	22	24
Depth of the water (feet)	0	7	8	2	0

A scientist measures the depth of the Doe River at Picnic Point. The river is 24 feet wide at this location. The measurements are taken in a straight line perpendicular to the edge of the river. The data are shown in the table above. The velocity of the water at Picnic Point, in feet per minute, is modeled by $v(t) = 16 + 2\sin\left(\sqrt{t+10}\right)$ for $0 \le t \le 120$ minutes.

(a) Use a trapezoidal sum with the four subintervals indicated by the data in the table to approximate the area of the cross section of the river at Picnic Point, in square feet. Show the computations that lead to your answer.

(b) The volumetric flow at a location along the river is the product of the cross-sectional area and the velocity of the water at that location. Use your approximation from part (a) to estimate the average value of the volumetric flow at Picnic Point, in cubic feet per minute, from $t = 0$ to $t = 120$ minutes.

(c) The scientist proposes the function f, given by $f(x) = 8\sin\left(\frac{\pi x}{24}\right)$, as a model for the depth of the water, in feet, at Picnic Point x feet from the river's edge. Find the area of the cross section of the river at Picnic Point based on this model.

(d) Recall that the volumetric flow is the product of the cross-sectional area and the velocity of the water at a location. To prevent flooding, water must be diverted if the average value of the volumetric flow at Picnic Point exceeds 2100 cubic feet per minute for a 20-minute period. Using your answer from part (c), find the average value of the volumetric flow during the time interval $40 \le t \le 60$ minutes. Does this value indicate that the water must be diverted?

(a) $\frac{(0+7)}{2}\cdot 8 + \frac{(7+8)}{2}\cdot 6 + \frac{(8+2)}{2}\cdot 8 + \frac{(2+0)}{2}\cdot 2$
$= 115 \text{ ft}^2$

1 : trapezoidal approximation

(b) $\frac{1}{120}\int_0^{120} 115v(t)\,dt$
$= 1807.169$ or $1807.170 \text{ ft}^3/\text{min}$

3 : { 1 : limits and average value constant; 1 : integrand; 1 : answer }

(c) $\int_0^{24} 8\sin\left(\frac{\pi x}{24}\right)dx = 122.230$ or 122.231 ft^2

2 : { 1 : integral; 1 : answer }

(d) Let C be the cross-sectional area approximation from part (c). The average volumetric flow is
$\frac{1}{20}\int_{40}^{60} C \cdot v(t)\,dt = 2181.912$ or $2181.913 \text{ ft}^3/\text{min}$.

Yes, water must be diverted since the average volumetric flow for this 20-minute period exceeds $2100 \text{ ft}^3/\text{min}$.

3 : { 1 : volumetric flow integral; 1 : average volumetric flow; 1 : answer with reason }

Question 4

Let f be the function given by $f(x) = kx^2 - x^3$, where k is a positive constant. Let R be the region in the first quadrant bounded by the graph of f and the x-axis.

(a) Find all values of the constant k for which the area of R equals 2.

(b) For $k > 0$, write, but do not evaluate, an integral expression in terms of k for the volume of the solid generated when R is rotated about the x-axis.

(c) For $k > 0$, write, but do not evaluate, an expression in terms of k, involving one or more integrals, that gives the perimeter of R.

(a) For $x \ge 0$, $f(x) = x^2(k - x) \ge 0$ if $0 \le x \le k$

$$\int_0^k \left(kx^2 - x^3\right) dx = \left(\frac{k}{3}x^3 - \frac{1}{4}x^4\right)\Bigg|_{x=0}^{x=k} = \frac{k^4}{12}$$

Area $= \dfrac{k^4}{12} = 2;\ k = \sqrt[4]{24}$

$4: \begin{cases} 1: \text{integral} \\ 1: \text{antiderivative} \\ 1: \text{value of integral} \\ 1: \text{answer} \end{cases}$

(b) Volume $= \pi \int_0^k \left(kx^2 - x^3\right)^2 dx$

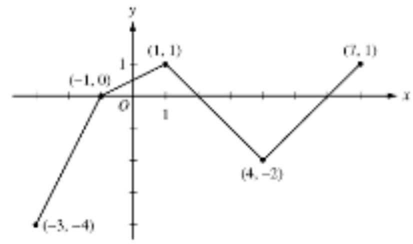
$2: \begin{cases} 1: \text{integrand} \\ 1: \text{limits and constant} \end{cases}$

(c) Perimeter $= k + \int_0^k \sqrt{1 + \left(2kx - 3x^2\right)^2}\, dx$

$3: \begin{cases} 1: \int_0^k \sqrt{1 + \left(f'(x)\right)^2}\, dx \\ 1: \text{uses } f'(x) = 2kx - 3x^2 \text{ in integrand} \\ 1: \text{answer} \end{cases}$

Question 5

Let g be a continuous function with $g(2) = 5$. The graph of the piecewise-linear function g', the derivative of g, is shown above for $-3 \le x \le 7$.



Graph of g'

(a) Find the x-coordinate of all points of inflection of the graph of $y = g(x)$ for $-3 < x < 7$. Justify your answer.

(b) Find the absolute maximum value of g on the interval $-3 \le x \le 7$. Justify your answer.

(c) Find the average rate of change of $g(x)$ on the interval $-3 \le x \le 7$.

(d) Find the average rate of change of $g'(x)$ on the interval $-3 \le x \le 7$. Does the Mean Value Theorem applied on the interval $-3 \le x \le 7$ guarantee a value of c, for $-3 < c < 7$, such that $g''(c)$ is equal to this average rate of change? Why or why not?

(a) g' changes from increasing to decreasing at $x = 1$;
g' changes from decreasing to increasing at $x = 4$.

Points of inflection for the graph of $y = g(x)$ occur at $x = 1$ and $x = 4$.

$2: \begin{cases} 1: x\text{-values} \\ 1: \text{justification} \end{cases}$

(b) The only sign change of g' from positive to negative in the interval is at $x = 2$.

$$g(-3) = 5 + \int_2^{-3} g'(x)\,dx = 5 + \left(-\frac{3}{2}\right) + 4 = \frac{15}{2}$$
$$g(2) = 5$$
$$g(7) = 5 + \int_2^{7} g'(x)\,dx = 5 + (-4) + \frac{1}{2} = \frac{3}{2}$$

The maximum value of g for $-3 \le x \le 7$ is $\frac{15}{2}$.

$3: \begin{cases} 1: \text{identifies } x = 2 \text{ as a candidate} \\ 1: \text{considers endpoints} \\ 1: \text{maximum value and justification} \end{cases}$

(c) $$\frac{g(7) - g(-3)}{7 - (-3)} = \frac{\frac{3}{2} - \frac{15}{2}}{10} = -\frac{3}{5}$$

$2: \begin{cases} 1: \text{difference quotient} \\ 1: \text{answer} \end{cases}$

(d) $$\frac{g'(7) - g'(-3)}{7 - (-3)} = \frac{1 - (-4)}{10} = \frac{1}{2}$$

No, the MVT does *not* guarantee the existence of a value c with the stated properties because g' is not differentiable for at least one point in $-3 < x < 7$.

$2: \begin{cases} 1: \text{average value of } g'(x) \\ 1: \text{answer "No" with reason} \end{cases}$

Question 6

Let f be the function given by $f(x) = \dfrac{2x}{1+x^2}$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 0$.

(b) Does the series found in part (a), when evaluated at $x = 1$, converge to $f(1)$? Explain why or why not.

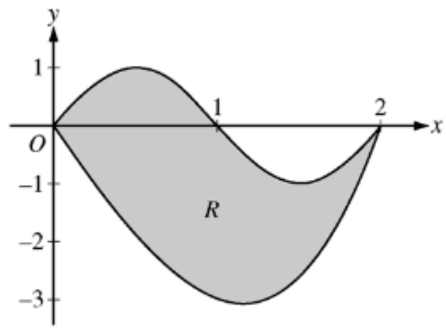
(c) The derivative of $\ln(1+x^2)$ is $\dfrac{2x}{1+x^2}$. Write the first four nonzero terms of the Taylor series for $\ln(1+x^2)$ about $x = 0$.

(d) Use the series found in part (c) to find a rational number A such that $\left|A - \ln\left(\frac{5}{4}\right)\right| < \frac{1}{100}$. Justify your answer.

Solution	Scoring
(a) $\dfrac{1}{1-u} = 1 + u + u^2 + \cdots + u^n + \cdots$ $\dfrac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + \left(-x^2\right)^n + \cdots$ $\dfrac{2x}{1+x^2} = 2x - 2x^3 + 2x^5 - 2x^7 + \cdots + (-1)^n 2x^{2n+1} + \cdots$	3 : { 1 : two of the first four terms 1 : remaining terms 1 : general term
(b) No, the series does not converge when $x = 1$ because when $x = 1$, the terms of the series do not converge to 0.	1 : answer with reason
(c) $\ln\left(1+x^2\right) = \int_0^x \dfrac{2t}{1+t^2}\,dt$ $= \int_0^x \left(2t - 2t^3 + 2t^5 - 2t^7 + \cdots\right) dt$ $= x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 - \frac{1}{4}x^8 + \cdots$	2 : { 1 : two of the first four terms 1 : remaining terms
(d) $\ln\left(\frac{5}{4}\right) = \ln\left(1 + \frac{1}{4}\right) = \left(\frac{1}{2}\right)^2 - \frac{1}{2}\left(\frac{1}{2}\right)^4 + \frac{1}{3}\left(\frac{1}{2}\right)^6 - \frac{1}{4}\left(\frac{1}{2}\right)^8 + \cdots$ Let $A = \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^4 = \frac{7}{32}$. Since the series is a converging alternating series and the absolute values of the individual terms decrease to 0, $\left|A - \ln\left(\frac{5}{4}\right)\right| < \left|\frac{1}{3}\left(\frac{1}{2}\right)^6\right| = \frac{1}{3} \cdot \frac{1}{64} < \frac{1}{100}$.	3 : { 1 : uses $x = \frac{1}{2}$ 1 : value of A 1 : justification

CALCULUS BC
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.



1. Let R be the region bounded by the graphs of $y = \sin(\pi x)$ and $y = x^3 - 4x$, as shown in the figure above.
   (a) Find the area of R.
   (b) The horizontal line $y = -2$ splits the region R into two parts. Write, but do not evaluate, an integral expression for the area of the part of R that is below this horizontal line.
   (c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of this solid.
   (d) The region R models the surface of a small pond. At all points in R at a distance x from the y-axis, the depth of the water is given by $h(x) = 3 - x$. Find the volume of water in the pond.

---

t (hours)	0	1	3	4	7	8	9
$L(t)$ (people)	120	156	176	126	150	80	0

2. Concert tickets went on sale at noon $(t = 0)$ and were sold out within 9 hours. The number of people waiting in line to purchase tickets at time t is modeled by a twice-differentiable function L for $0 \le t \le 9$. Values of $L(t)$ at various times t are shown in the table above.
   (a) Use the data in the table to estimate the rate at which the number of people waiting in line was changing at 5:30 P.M. $(t = 5.5)$. Show the computations that lead to your answer. Indicate units of measure.
   (b) Use a trapezoidal sum with three subintervals to estimate the average number of people waiting in line during the first 4 hours that tickets were on sale.
   (c) For $0 \le t \le 9$, what is the fewest number of times at which $L'(t)$ must equal 0 ? Give a reason for your answer.
   (d) The rate at which tickets were sold for $0 \le t \le 9$ is modeled by $r(t) = 550te^{-t/2}$ tickets per hour. Based on the model, how many tickets were sold by 3 P.M. $(t = 3)$, to the nearest whole number?

---

x	$h(x)$	$h'(x)$	$h''(x)$	$h'''(x)$	$h^{(4)}(x)$
1	11	30	42	99	18
2	80	128	$\frac{488}{3}$	$\frac{448}{3}$	$\frac{584}{9}$
3	317	$\frac{753}{2}$	$\frac{1383}{4}$	$\frac{3483}{16}$	$\frac{1125}{16}$

3. Let h be a function having derivatives of all orders for $x > 0$. Selected values of h and its first four derivatives are indicated in the table above. The function h and these four derivatives are increasing on the interval $1 \le x \le 3$.

   (a) Write the first-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$. Is this approximation greater than or less than $h(1.9)$? Explain your reasoning.

   (b) Write the third-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$.

   (c) Use the Lagrange error bound to show that the third-degree Taylor polynomial for h about $x = 2$ approximates $h(1.9)$ with error less than 3×10^{-4}.

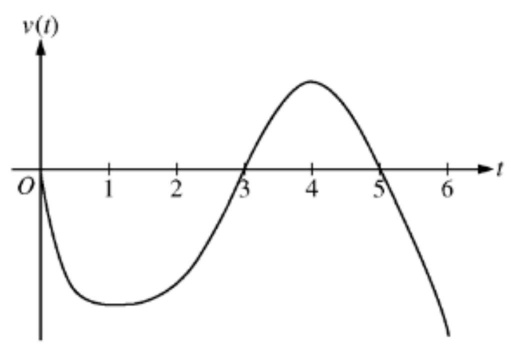
CALCULUS BC

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.



Graph of v

4. A particle moves along the x-axis so that its velocity at time t, for $0 \le t \le 6$, is given by a differentiable function v whose graph is shown above. The velocity is 0 at $t = 0$, $t = 3$, and $t = 5$, and the graph has horizontal tangents at $t = 1$ and $t = 4$. The areas of the regions bounded by the t-axis and the graph of v on the intervals $[0, 3]$, $[3, 5]$, and $[5, 6]$ are 8, 3, and 2, respectively. At time $t = 0$, the particle is at $x = -2$.

   (a) For $0 \le t \le 6$, find both the time and the position of the particle when the particle is farthest to the left. Justify your answer.

   (b) For how many values of t, where $0 \le t \le 6$, is the particle at $x = -8$? Explain your reasoning.

   (c) On the interval $2 < t < 3$, is the speed of the particle increasing or decreasing? Give a reason for your answer.

   (d) During what time intervals, if any, is the acceleration of the particle negative? Justify your answer.

5. The derivative of a function f is given by $f'(x) = (x - 3)e^x$ for $x > 0$, and $f(1) = 7$.

(a) The function f has a critical point at $x = 3$. At this point, does f have a relative minimum, a relative maximum, or neither? Justify your answer.

(b) On what intervals, if any, is the graph of f both decreasing and concave up? Explain your reasoning.

(c) Find the value of $f(3)$.

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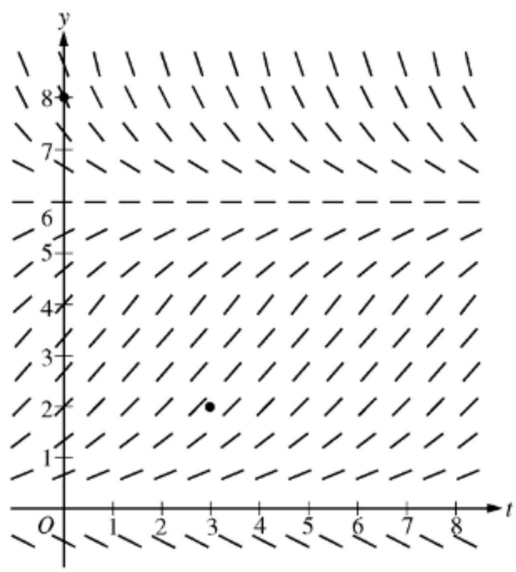
6. Consider the logistic differential equation $\frac{dy}{dt} = \frac{y}{8}(6 - y)$. Let $y = f(t)$ be the particular solution to the differential equation with $f(0) = 8$.

(a) A slope field for this differential equation is given below. Sketch possible solution curves through the points $(3, 2)$ and $(0, 8)$.

(Note: Use the axes provided in the exam booklet.)

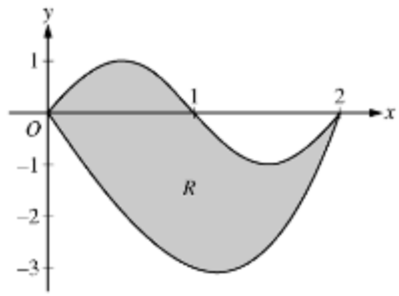


(b) Use Euler's method, starting at $t = 0$ with two steps of equal size, to approximate $f(1)$.

(c) Write the second-degree Taylor polynomial for f about $t = 0$, and use it to approximate $f(1)$.

(d) What is the range of f for $t \geq 0$?

Question 1



Let R be the region bounded by the graphs of $y = \sin(\pi x)$ and $y = x^3 - 4x$, as shown in the figure above.

(a) Find the area of R.

(b) The horizontal line $y = -2$ splits the region R into two parts. Write, but do not evaluate, an integral expression for the area of the part of R that is below this horizontal line.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of this solid.

(d) The region R models the surface of a small pond. At all points in R at a distance x from the y-axis, the depth of the water is given by $h(x) = 3 - x$. Find the volume of water in the pond.

(a) $\sin(\pi x) = x^3 - 4x$ at $x = 0$ and $x = 2$ Area $= \int_0^2 \left(\sin(\pi x) - \left(x^3 - 4x\right)\right) dx = 4$	$3 : \begin{cases} 1 : \text{limits} \\ 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$
(b) $x^3 - 4x = -2$ at $r = 0.5391889$ and $s = 1.6751309$ The area of the stated region is $\int_r^s \left(-2 - \left(x^3 - 4x\right)\right) dx$	$2 : \begin{cases} 1 : \text{limits} \\ 1 : \text{integrand} \end{cases}$
(c) Volume $= \int_0^2 \left(\sin(\pi x) - \left(x^3 - 4x\right)\right)^2 dx = 9.978$	$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$
(d) Volume $= \int_0^2 (3 - x)\left(\sin(\pi x) - \left(x^3 - 4x\right)\right) dx = 8.369$ or 8.370	$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

Question 2

t (hours)	0	1	3	4	7	8	9
$L(t)$ (people)	120	156	176	126	150	80	0

Concert tickets went on sale at noon $(t = 0)$ and were sold out within 9 hours. The number of people waiting in line to purchase tickets at time t is modeled by a twice-differentiable function L for $0 \le t \le 9$. Values of $L(t)$ at various times t are shown in the table above.

(a) Use the data in the table to estimate the rate at which the number of people waiting in line was changing at 5:30 P.M. $(t = 5.5)$. Show the computations that lead to your answer. Indicate units of measure.

(b) Use a trapezoidal sum with three subintervals to estimate the average number of people waiting in line during the first 4 hours that tickets were on sale.

(c) For $0 \le t \le 9$, what is the fewest number of times at which $L'(t)$ must equal 0? Give a reason for your answer.

(d) The rate at which tickets were sold for $0 \le t \le 9$ is modeled by $r(t) = 550te^{-t/2}$ tickets per hour. Based on the model, how many tickets were sold by 3 P.M. $(t = 3)$, to the nearest whole number?

(a) $L'(5.5) \approx \dfrac{L(7) - L(4)}{7 - 4} = \dfrac{150 - 126}{3} = 8$ people per hour

$2 : \begin{cases} 1 : \text{estimate} \\ 1 : \text{units} \end{cases}$

(b) The average number of people waiting in line during the first 4 hours is approximately

$\dfrac{1}{4}\left(\dfrac{L(0)+L(1)}{2}(1-0) + \dfrac{L(1)+L(3)}{2}(3-1) + \dfrac{L(3)+L(4)}{2}(4-3)\right)$

$= 155.25$ people

$2 : \begin{cases} 1 : \text{trapezoidal sum} \\ 1 : \text{answer} \end{cases}$

(c) L is differentiable on $[0, 9]$ so the Mean Value Theorem implies $L'(t) > 0$ for some t in $(1, 3)$ and some t in $(4, 7)$. Similarly, $L'(t) < 0$ for some t in $(3, 4)$ and some t in $(7, 8)$. Then, since L' is continuous on $[0, 9]$, the Intermediate Value Theorem implies that $L'(t) = 0$ for at least three values of t in $[0, 9]$.

$3 : \begin{cases} 1 : \text{considers change in sign of } L' \\ 1 : \text{analysis} \\ 1 : \text{conclusion} \end{cases}$

OR

The continuity of L on $[1, 4]$ implies that L attains a maximum value there. Since $L(3) > L(1)$ and $L(3) > L(4)$, this maximum occurs on $(1, 4)$. Similarly, L attains a minimum on $(3, 7)$ and a maximum on $(4, 8)$. L is differentiable, so $L'(t) = 0$ at each relative extreme point on $(0, 9)$. Therefore $L'(t) = 0$ for at least three values of t in $[0, 9]$.

OR

$3 : \begin{cases} 1 : \text{considers relative extrema of } L \text{ on } (0, 9) \\ 1 : \text{analysis} \\ 1 : \text{conclusion} \end{cases}$

[*Note: There is a function L that satisfies the given conditions with $L'(t) = 0$ for exactly three values of t.*]

(d) $\int_0^3 r(t)\,dt = 972.784$

There were approximately 973 tickets sold by 3 P.M.

$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits and answer} \end{cases}$

Question 3

x	$h(x)$	$h'(x)$	$h''(x)$	$h'''(x)$	$h^{(4)}(x)$
1	11	30	42	99	18
2	80	128	$\frac{488}{3}$	$\frac{448}{3}$	$\frac{584}{9}$
3	317	$\frac{753}{2}$	$\frac{1383}{4}$	$\frac{3483}{16}$	$\frac{1125}{16}$

Let h be a function having derivatives of all orders for $x > 0$. Selected values of h and its first four derivatives are indicated in the table above. The function h and these four derivatives are increasing on the interval $1 \le x \le 3$.

(a) Write the first-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$. Is this approximation greater than or less than $h(1.9)$? Explain your reasoning.

(b) Write the third-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$.

(c) Use the Lagrange error bound to show that the third-degree Taylor polynomial for h about $x = 2$ approximates $h(1.9)$ with error less than 3×10^{-4}.

(a) $P_1(x) = 80 + 128(x-2)$, so $h(1.9) \approx P_1(1.9) = 67.2$

$P_1(1.9) < h(1.9)$ since h' is increasing on the interval $1 \le x \le 3$.

$4: \begin{cases} 2: P_1(x) \\ 1: P_1(1.9) \\ 1: P_1(1.9) < h(1.9) \text{ with reason} \end{cases}$

(b) $P_3(x) = 80 + 128(x-2) + \frac{488}{6}(x-2)^2 + \frac{448}{18}(x-2)^3$

$h(1.9) \approx P_3(1.9) = 67.988$

$3: \begin{cases} 2: P_3(x) \\ 1: P_3(1.9) \end{cases}$

(c) The fourth derivative of h is increasing on the interval $1 \le x \le 3$, so $\max_{1.9 \le x \le 2} \left| h^{(4)}(x) \right| = \frac{584}{9}$.

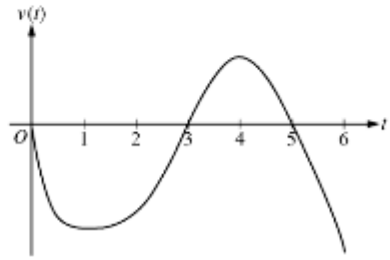
Therefore, $|h(1.9) - P_3(1.9)| \le \frac{584}{9} \frac{|1.9-2|^4}{4!}$

$= 2.7037 \times 10^{-4}$

$< 3 \times 10^{-4}$

$2: \begin{cases} 1: \text{form of Lagrange error estimate} \\ 1: \text{reasoning} \end{cases}$

Question 4



Graph of v

A particle moves along the x-axis so that its velocity at time t, for $0 \le t \le 6$, is given by a differentiable function v whose graph is shown above. The velocity is 0 at $t = 0$, $t = 3$, and $t = 5$, and the graph has horizontal tangents at $t = 1$ and $t = 4$. The areas of the regions bounded by the t-axis and the graph of v on the intervals $[0, 3]$, $[3, 5]$, and $[5, 6]$ are 8, 3, and 2, respectively. At time $t = 0$, the particle is at $x = -2$.

(a) For $0 \le t \le 6$, find both the time and the position of the particle when the particle is farthest to the left. Justify your answer.

(b) For how many values of t, where $0 \le t \le 6$, is the particle at $x = -8$? Explain your reasoning.

(c) On the interval $2 < t < 3$, is the speed of the particle increasing or decreasing? Give a reason for your answer.

(d) During what time intervals, if any, is the acceleration of the particle negative? Justify your answer.

(a) Since $v(t) < 0$ for $0 < t < 3$ and $5 < t < 6$, and $v(t) > 0$ for $3 < t < 5$, we consider $t = 3$ and $t = 6$. $x(3) = -2 + \int_0^3 v(t)\,dt = -2 - 8 = -10$ $x(6) = -2 + \int_0^6 v(t)\,dt = -2 - 8 + 3 - 2 = -9$ Therefore, the particle is farthest left at time $t = 3$ when its position is $x(3) = -10$.	3 : { 1 : identifies $t = 3$ as a candidate 1 : considers $\int_0^6 v(t)\,dt$ 1 : conclusion
(b) The particle moves continuously and monotonically from $x(0) = -2$ to $x(3) = -10$. Similarly, the particle moves continuously and monotonically from $x(3) = -10$ to $x(5) = -7$ and also from $x(5) = -7$ to $x(6) = -9$. By the Intermediate Value Theorem, there are three values of t for which the particle is at $x(t) = -8$.	3 : { 1 : positions at $t = 3$, $t = 5$, and $t = 6$ 1 : description of motion 1 : conclusion
(c) The speed is decreasing on the interval $2 < t < 3$ since on this interval $v < 0$ and v is increasing.	1 : answer with reason
(d) The acceleration is negative on the intervals $0 < t < 1$ and $4 < t < 6$ since velocity is decreasing on these intervals.	2 : { 1 : answer 1 : justification

Question 5

The derivative of a function f is given by $f'(x) = (x-3)e^x$ for $x > 0$, and $f(1) = 7$.

(a) The function f has a critical point at $x = 3$. At this point, does f have a relative minimum, a relative maximum, or neither? Justify your answer.

(b) On what intervals, if any, is the graph of f both decreasing and concave up? Explain your reasoning.

(c) Find the value of $f(3)$.

(a) $f'(x) < 0$ for $0 < x < 3$ and $f'(x) > 0$ for $x > 3$

Therefore, f has a relative minimum at $x = 3$.

$2: \begin{cases} 1\text{: minimum at } x = 3 \\ 1\text{: justification} \end{cases}$

(b) $f''(x) = e^x + (x-3)e^x = (x-2)e^x$

$f''(x) > 0$ for $x > 2$

$f'(x) < 0$ for $0 < x < 3$

Therefore, the graph of f is both decreasing and concave up on the interval $2 < x < 3$.

$3: \begin{cases} 2: f''(x) \\ 1\text{ : answer with reason} \end{cases}$

(c) $f(3) = f(1) + \int_1^3 f'(x)\,dx = 7 + \int_1^3 (x-3)e^x\,dx$

$$u = x - 3 \quad dv = e^x\,dx$$
$$du = dx \quad v = e^x$$

$$f(3) = 7 + (x-3)e^x\Big|_1^3 - \int_1^3 e^x\,dx$$

$$= 7 + \left((x-3)e^x - e^x\right)\Big|_1^3$$

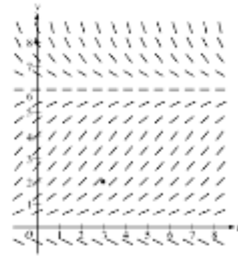
$$= 7 + 3e - e^3$$

$4: \begin{cases} 1\text{ : uses initial condition} \\ 2\text{ : integration by parts} \\ 1\text{ : answer} \end{cases}$

Question 6

Consider the logistic differential equation $\frac{dy}{dt} = \frac{y}{8}(6 - y)$. Let $y = f(t)$ be the particular solution to the differential equation with $f(0) = 8$.

(a) A slope field for this differential equation is given below. Sketch possible solution curves through the points $(3, 2)$ and $(0, 8)$.
(Note: Use the axes provided in the exam booklet.)

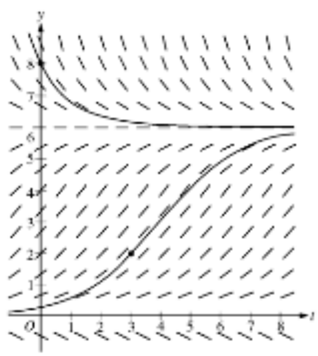


(b) Use Euler's method, starting at $t = 0$ with two steps of equal size, to approximate $f(1)$.

(c) Write the second-degree Taylor polynomial for f about $t = 0$, and use it to approximate $f(1)$.

(d) What is the range of f for $t \geq 0$?

(a)



$2 : \begin{cases} 1\text{: solution curve through } (0,8) \\ 1\text{: solution curve through } (3,2) \end{cases}$

(b) $f\left(\frac{1}{2}\right) \approx 8 + (-2)\left(\frac{1}{2}\right) = 7$

$f(1) \approx 7 + \left(-\frac{7}{8}\right)\left(\frac{1}{2}\right) = \frac{105}{16}$

$2 : \begin{cases} 1 : \text{Euler's method with two steps} \\ 1 : \text{approximation of } f(1) \end{cases}$

(c) $\frac{d^2y}{dt^2} = \frac{1}{8}\frac{dy}{dt}(6 - y) + \frac{y}{8}\left(-\frac{dy}{dt}\right)$

$f(0) = 8$; $f'(0) = \left.\frac{dy}{dt}\right|_{t=0} = \frac{8}{8}(6 - 8) = -2$; and

$f''(0) = \left.\frac{d^2y}{dt^2}\right|_{t=0} = \frac{1}{8}(-2)(-2) + \frac{8}{8}(2) = \frac{5}{2}$

The second-degree Taylor polynomial for f about $t = 0$ is $P_2(t) = 8 - 2t + \frac{5}{4}t^2$.

$f(1) \approx P_2(1) = \frac{29}{4}$

$4 : \begin{cases} 2 : \frac{d^2y}{dt^2} \\ 1 : \text{second-degree Taylor polynomial} \\ 1 : \text{approximation of } f(1) \end{cases}$

(d) The range of f for $t \geq 0$ is $6 < y \leq 8$.

1 : answer

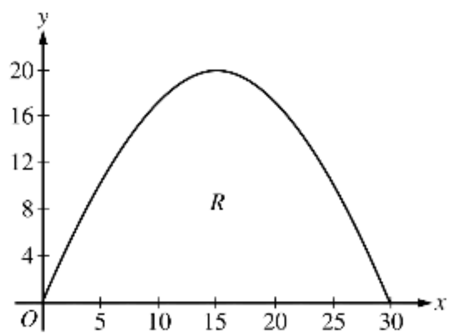
CALCULUS BC

SECTION II, Part A

Time—45 minutes

Number of problems—3

A graphing calculator is required for some problems or parts of problems.



1. A baker is creating a birthday cake. The base of the cake is the region R in the first quadrant under the graph of $y = f(x)$ for $0 \le x \le 30$, where $f(x) = 20\sin\left(\frac{\pi x}{30}\right)$. Both x and y are measured in centimeters. The region R is shown in the figure above. The derivative of f is $f'(x) = \frac{2\pi}{3}\cos\left(\frac{\pi x}{30}\right)$.

   (a) The region R is cut out of a 30-centimeter-by-20-centimeter rectangular sheet of cardboard, and the remaining cardboard is discarded. Find the area of the discarded cardboard.

   (b) The cake is a solid with base R. Cross sections of the cake perpendicular to the x-axis are semicircles. If the baker uses 0.05 gram of unsweetened chocolate for each cubic centimeter of cake, how many grams of unsweetened chocolate will be in the cake?

   (c) Find the perimeter of the base of the cake.

---

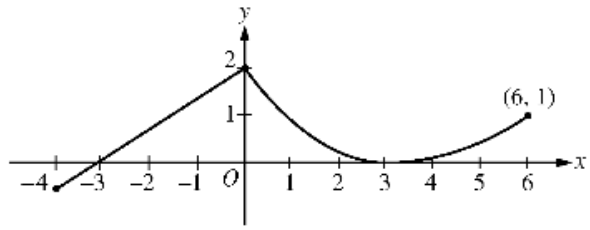
2. A storm washed away sand from a beach, causing the edge of the water to get closer to a nearby road. The rate at which the distance between the road and the edge of the water was changing during the storm is modeled by $f(t) = \sqrt{t} + \cos t - 3$ meters per hour, t hours after the storm began. The edge of the water was 35 meters from the road when the storm began, and the storm lasted 5 hours. The derivative of $f(t)$ is $f'(t) = \frac{1}{2\sqrt{t}} - \sin t$.

   (a) What was the distance between the road and the edge of the water at the end of the storm?

   (b) Using correct units, interpret the value $f'(4) = 1.007$ in terms of the distance between the road and the edge of the water.

   (c) At what time during the 5 hours of the storm was the distance between the road and the edge of the water decreasing most rapidly? Justify your answer.

   (d) After the storm, a machine pumped sand back onto the beach so that the distance between the road and the edge of the water was growing at a rate of $g(p)$ meters per day, where p is the number of days since pumping began. Write an equation involving an integral expression whose solution would give the number of days that sand must be pumped to restore the original distance between the road and the edge of the water.



Graph of f

3. A continuous function f is defined on the closed interval $-4 \le x \le 6$. The graph of f consists of a line segment and a curve that is tangent to the x-axis at $x = 3$, as shown in the figure above. On the interval $0 < x < 6$, the function f is twice differentiable, with $f''(x) > 0$.

(a) Is f differentiable at $x = 0$? Use the definition of the derivative with one-sided limits to justify your answer.

(b) For how many values of a, $-4 \le a < 6$, is the average rate of change of f on the interval $[a, 6]$ equal to 0 ? Give a reason for your answer.

(c) Is there a value of a, $-4 \le a < 6$, for which the Mean Value Theorem, applied to the interval $[a, 6]$, guarantees a value c, $a < c < 6$, at which $f'(c) = \frac{1}{3}$? Justify your answer.

(d) The function g is defined by $g(x) = \int_0^x f(t)\,dt$ for $-4 \le x \le 6$. On what intervals contained in $[-4, 6]$ is the graph of g concave up? Explain your reasoning.

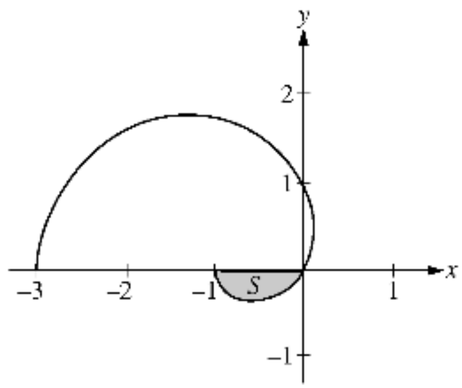
CALCULUS BC

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.

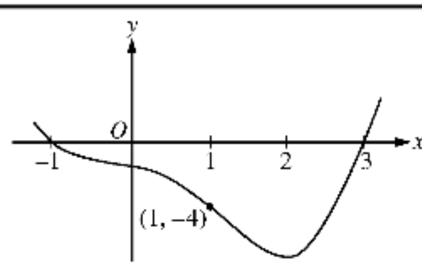


4. The graph of the polar curve $r = 1 - 2\cos\theta$ for $0 \le \theta \le \pi$ is shown above. Let S be the shaded region in the third quadrant bounded by the curve and the x-axis.

   (a) Write an integral expression for the area of S.

   (b) Write expressions for $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$ in terms of θ.

   (c) Write an equation in terms of x and y for the line tangent to the graph of the polar curve at the point where $\theta = \frac{\pi}{2}$. Show the computations that lead to your answer.



Graph of f'

5. Let f be a twice-differentiable function defined on the interval $-1.2 < x < 3.2$ with $f(1) = 2$. The graph of f', the derivative of f, is shown above. The graph of f' crosses the x-axis at $x = -1$ and $x = 3$ and has a horizontal tangent at $x = 2$. Let g be the function given by $g(x) = e^{f(x)}$.

   (a) Write an equation for the line tangent to the graph of g at $x = 1$.

   (b) For $-1.2 < x < 3.2$, find all values of x at which g has a local maximum. Justify your answer.

   (c) The second derivative of g is $g''(x) = e^{f(x)}\left[\left(f'(x)\right)^2 + f''(x)\right]$. Is $g''(-1)$ positive, negative, or zero? Justify your answer.

   (d) Find the average rate of change of g', the derivative of g, over the interval $[1, 3]$.

---

6. The function f is defined by the power series

$$f(x) = 1 + (x+1) + (x+1)^2 + \cdots + (x+1)^n + \cdots = \sum_{n=0}^{\infty} (x+1)^n$$

for all real numbers x for which the series converges.

(a) Find the interval of convergence of the power series for f. Justify your answer.

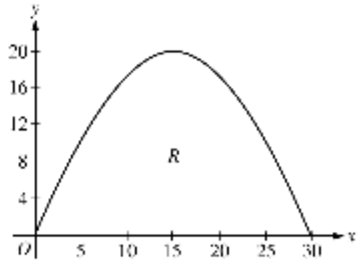
(b) The power series above is the Taylor series for f about $x = -1$. Find the sum of the series for f.

(c) Let g be the function defined by $g(x) = \int_{-1}^{x} f(t)\,dt$. Find the value of $g\left(-\frac{1}{2}\right)$, if it exists, or explain why $g\left(-\frac{1}{2}\right)$ cannot be determined.

(d) Let h be the function defined by $h(x) = f\left(x^2 - 1\right)$. Find the first three nonzero terms and the general term of the Taylor series for h about $x = 0$, and find the value of $h\left(\frac{1}{2}\right)$.

Question 1

A baker is creating a birthday cake. The base of the cake is the region R in the first quadrant under the graph of $y = f(x)$ for $0 \le x \le 30$, where $f(x) = 20\sin\left(\frac{\pi x}{30}\right)$. Both x and y are measured in centimeters. The region R is shown in the figure above. The derivative of f is $f'(x) = \frac{2\pi}{3}\cos\left(\frac{\pi x}{30}\right)$.



(a) The region R is cut out of a 30-centimeter-by-20-centimeter rectangular sheet of cardboard, and the remaining cardboard is discarded. Find the area of the discarded cardboard.

(b) The cake is a solid with base R. Cross sections of the cake perpendicular to the x-axis are semicircles. If the baker uses 0.05 gram of unsweetened chocolate for each cubic centimeter of cake, how many grams of unsweetened chocolate will be in the cake?

(c) Find the perimeter of the base of the cake.

(a) Area $= 30 \cdot 20 - \int_0^{30} f(x)\,dx = 218.028 \text{ cm}^2$	$3: \begin{cases} 2: \text{integral} \\ 1: \text{answer} \end{cases}$
(b) Volume $= \int_0^{30} \frac{\pi}{2}\left(\frac{f(x)}{2}\right)^2 dx = 2356.194 \text{ cm}^3$ Therefore, the baker needs $2356.194 \times 0.05 = 117.809$ or 117.810 grams of chocolate.	$3: \begin{cases} 2: \text{integral} \\ 1: \text{answer} \end{cases}$
(c) Perimeter $= 30 + \int_0^{30} \sqrt{1 + (f'(x))^2}\,dx = 81.803$ or 81.804 cm	$3: \begin{cases} 2: \text{integral} \\ 1: \text{answer} \end{cases}$

Question 2

A storm washed away sand from a beach, causing the edge of the water to get closer to a nearby road. The rate at which the distance between the road and the edge of the water was changing during the storm is modeled by $f(t) = \sqrt{t} + \cos t - 3$ meters per hour, t hours after the storm began. The edge of the water was 35 meters from the road when the storm began, and the storm lasted 5 hours. The derivative of $f(t)$ is $f'(t) = \dfrac{1}{2\sqrt{t}} - \sin t$.

(a) What was the distance between the road and the edge of the water at the end of the storm?

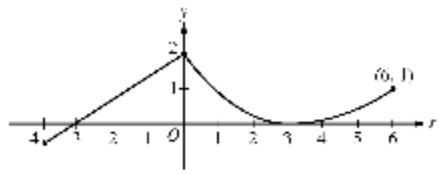
(b) Using correct units, interpret the value $f'(4) = 1.007$ in terms of the distance between the road and the edge of the water.

(c) At what time during the 5 hours of the storm was the distance between the road and the edge of the water decreasing most rapidly? Justify your answer.

(d) After the storm, a machine pumped sand back onto the beach so that the distance between the road and the edge of the water was growing at a rate of $g(p)$ meters per day, where p is the number of days since pumping began. Write an equation involving an integral expression whose solution would give the number of days that sand must be pumped to restore the original distance between the road and the edge of the water.

Solution	Points
(a) $35 + \int_0^5 f(t)\,dt = 26.494$ or 26.495 meters	$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$
(b) Four hours after the storm began, the rate of change of the distance between the road and the edge of the water is increasing at a rate of 1.007 meters / hours^2.	$2: \begin{cases} 1: \text{interpretation of } f'(4) \\ 1: \text{units} \end{cases}$
(c) $f'(t) = 0$ when $t = 0.66187$ and $t = 2.84038$ The minimum of f for $0 \le t \le 5$ may occur at 0, 0.66187, 2.84038, or 5. $f(0) = -2$ $f(0.66187) = -1.39760$ $f(2.84038) = -2.26963$ $f(5) = -0.48027$ The distance between the road and the edge of the water was decreasing most rapidly at time $t = 2.840$ hours after the storm began.	$3: \begin{cases} 1: \text{considers } f'(t) = 0 \\ 1: \text{answer} \\ 1: \text{justification} \end{cases}$
(d) $-\int_0^5 f(t)\,dt = \int_0^x g(p)\,dp$	$2: \begin{cases} 1: \text{integral of } g \\ 1: \text{answer} \end{cases}$

Question 3



Graph of f

A continuous function f is defined on the closed interval $-4 \le x \le 6$. The graph of f consists of a line segment and a curve that is tangent to the x-axis at $x = 3$, as shown in the figure above. On the interval $0 < x < 6$, the function f is twice differentiable, with $f''(x) > 0$.

(a) Is f differentiable at $x = 0$? Use the definition of the derivative with one-sided limits to justify your answer.

(b) For how many values of a, $-4 \le a < 6$, is the average rate of change of f on the interval $[a, 6]$ equal to 0? Give a reason for your answer.

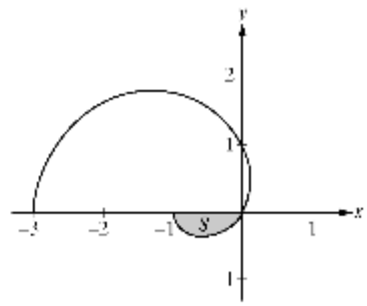
(c) Is there a value of a, $-4 \le a < 6$, for which the Mean Value Theorem, applied to the interval $[a, 6]$, guarantees a value c, $a < c < 6$, at which $f'(c) = \frac{1}{3}$? Justify your answer.

(d) The function g is defined by $g(x) = \int_0^x f(t)\,dt$ for $-4 \le x \le 6$. On what intervals contained in $[-4, 6]$ is the graph of g concave up? Explain your reasoning.

(a) $\lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \frac{2}{3}$ $\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} < 0$ Since the one-sided limits do not agree, f is not differentiable at $x = 0$.	$2: \begin{cases} 1: \text{sets up difference quotient at } x = 0 \\ 1: \text{answer with justification} \end{cases}$
(b) $\frac{f(6) - f(a)}{6 - a} = 0$ when $f(a) = f(6)$. There are two values of a for which this is true.	$2: \begin{cases} 1: \text{expression for average rate of change} \\ 1: \text{answer with reason} \end{cases}$
(c) Yes, $a = 3$. The function f is differentiable on the interval $3 < x < 6$ and continuous on $3 \le x \le 6$. Also, $\frac{f(6) - f(3)}{6 - 3} = \frac{1 - 0}{6 - 3} = \frac{1}{3}$. By the Mean Value Theorem, there is a value c, $3 < c < 6$, such that $f'(c) = \frac{1}{3}$.	$2: \begin{cases} 1: \text{answers "yes" and identifies } a = 3 \\ 1: \text{justification} \end{cases}$
(d) $g'(x) = f(x)$, $g''(x) = f'(x)$ $g''(x) > 0$ when $f'(x) > 0$ This is true for $-4 < x < 0$ and $3 < x < 6$.	$3: \begin{cases} 1: g'(x) = f(x) \\ 1: \text{considers } g''(x) > 0 \\ 1: \text{answer} \end{cases}$

Question 4

The graph of the polar curve $r = 1 - 2\cos\theta$ for $0 \le \theta \le \pi$ is shown above. Let S be the shaded region in the third quadrant bounded by the curve and the x-axis.



(a) Write an integral expression for the area of S.

(b) Write expressions for $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$ in terms of θ.

(c) Write an equation in terms of x and y for the line tangent to the graph of the polar curve at the point where $\theta = \frac{\pi}{2}$. Show the computations that lead to your answer.

(a) $r(0) = -1$; $r(\theta) = 0$ when $\theta = \frac{\pi}{3}$.

Area of $S = \frac{1}{2}\int_0^{\pi/3} (1 - 2\cos\theta)^2\, d\theta$

$2: \begin{cases} 1 : \text{limits and constant} \\ 1 : \text{integrand} \end{cases}$

(b) $x = r\cos\theta$ and $y = r\sin\theta$

$\frac{dr}{d\theta} = 2\sin\theta$

$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta = 4\sin\theta\cos\theta - \sin\theta$

$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta = 2\sin^2\theta + (1 - 2\cos\theta)\cos\theta$

$4: \begin{cases} 1 : \text{uses } x = r\cos\theta \text{ and } y = r\sin\theta \\ 1 : \frac{dr}{d\theta} \\ 2 : \text{answer} \end{cases}$

(c) When $\theta = \frac{\pi}{2}$, we have $x = 0$, $y = 1$.

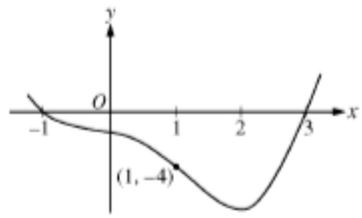
$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{2}} = \left.\frac{dy/d\theta}{dx/d\theta}\right|_{\theta=\frac{\pi}{2}} = -2$

The tangent line is given by $y = 1 - 2x$.

$3: \begin{cases} 1 : \text{values for } x \text{ and } y \\ 1 : \text{expression for } \frac{dy}{dx} \\ 1 : \text{tangent line equation} \end{cases}$

Question 5

Let f be a twice-differentiable function defined on the interval $-1.2 < x < 3.2$ with $f(1) = 2$. The graph of f', the derivative of f, is shown above. The graph of f' crosses the x-axis at $x = -1$ and $x = 3$ and has a horizontal tangent at $x = 2$. Let g be the function given by $g(x) = e^{f(x)}$.



Graph of f'

(a) Write an equation for the line tangent to the graph of g at $x = 1$.

(b) For $-1.2 < x < 3.2$, find all values of x at which g has a local maximum. Justify your answer.

(c) The second derivative of g is $g''(x) = e^{f(x)}\left[(f'(x))^2 + f''(x)\right]$. Is $g''(-1)$ positive, negative, or zero? Justify your answer.

(d) Find the average rate of change of g', the derivative of g, over the interval $[1, 3]$.

(a) $g(1) = e^{f(1)} = e^2$

$g'(x) = e^{f(x)}f'(x)$, $g'(1) = e^{f(1)}f'(1) = -4e^2$

The tangent line is given by $y = e^2 - 4e^2(x - 1)$.

$3: \begin{cases} 1: g'(x) \\ 1: g(1) \text{ and } g'(1) \\ 1: \text{tangent line equation} \end{cases}$

(b) $g'(x) = e^{f(x)}f'(x)$

$e^{f(x)} > 0$ for all x

So, g' changes from positive to negative only when f' changes from positive to negative. This occurs at $x = -1$ only. Thus, g has a local maximum at $x = -1$.

$2: \begin{cases} 1: \text{answer} \\ 1: \text{justification} \end{cases}$

(c) $g''(-1) = e^{f(-1)}\left[(f'(-1))^2 + f''(-1)\right]$

$e^{f(-1)} > 0$ and $f'(-1) = 0$

Since f' is decreasing on a neighborhood of -1, $f''(-1) < 0$. Therefore, $g''(-1) < 0$.

$2: \begin{cases} 1: \text{answer} \\ 1: \text{justification} \end{cases}$

(d) $\dfrac{g'(3) - g'(1)}{3 - 1} = \dfrac{e^{f(3)}f'(3) - e^{f(1)}f'(1)}{2} = 2e^2$

$2: \begin{cases} 1: \text{difference quotient} \\ 1: \text{answer} \end{cases}$

Question 6

The function f is defined by the power series

$$f(x) = 1 + (x+1) + (x+1)^2 + \cdots + (x+1)^n + \cdots = \sum_{n=0}^{\infty} (x+1)^n$$

for all real numbers x for which the series converges.

(a) Find the interval of convergence of the power series for f. Justify your answer.

(b) The power series above is the Taylor series for f about $x = -1$. Find the sum of the series for f.

(c) Let g be the function defined by $g(x) = \int_{-1}^{x} f(t)\,dt$. Find the value of $g\left(-\frac{1}{2}\right)$, if it exists, or explain why $g\left(-\frac{1}{2}\right)$ cannot be determined.

(d) Let h be the function defined by $h(x) = f\left(x^2 - 1\right)$. Find the first three nonzero terms and the general term of the Taylor series for h about $x = 0$, and find the value of $h\left(\frac{1}{2}\right)$.

Solution	Scoring
(a) The power series is geometric with ratio $(x+1)$. The series converges if and only if $\lvert x+1 \rvert < 1$. Therefore, the interval of convergence is $-2 < x < 0$.	$3: \begin{cases} 1: \text{identifies as geometric} \\ 1: \lvert x+1\rvert < 1 \\ 1: \text{interval of convergence} \end{cases}$
OR	OR
$\lim_{n\to\infty} \left\lvert \frac{(x+1)^{n+1}}{(x+1)^n} \right\rvert = \lvert x+1 \rvert < 1$ when $-2 < x < 0$ At $x = -2$, the series is $\sum_{n=0}^{\infty} (-1)^n$, which diverges since the terms do not converge to 0. At $x = 0$, the series is $\sum_{n=0}^{\infty} 1$, which similarly diverges. Therefore, the interval of convergence is $-2 < x < 0$.	$3: \begin{cases} 1: \text{sets up limit of ratio} \\ 1: \text{radius of convergence} \\ 1: \text{interval of convergence} \end{cases}$
(b) Since the series is geometric, $f(x) = \sum_{n=0}^{\infty} (x+1)^n = \frac{1}{1-(x+1)} = -\frac{1}{x}$ for $-2 < x < 0$.	1 : answer
(c) $g\left(-\frac{1}{2}\right) = \int_{-1}^{-\frac{1}{2}} -\frac{1}{x}\,dx = -\ln\lvert x \rvert \Big\rvert_{x=-1}^{x=-\frac{1}{2}} = \ln 2$	$2: \begin{cases} 1: \text{antiderivative} \\ 1: \text{value} \end{cases}$
(d) $h(x) = f\left(x^2 - 1\right) = 1 + x^2 + x^4 + \cdots + x^{2n} + \cdots$ $h\left(\frac{1}{2}\right) = f\left(-\frac{3}{4}\right) = \frac{4}{3}$	$3: \begin{cases} 1: \text{first three terms} \\ 1: \text{general term} \\ 1: \text{value of } h\left(\frac{1}{2}\right) \end{cases}$

2009 AP® CALCULUS BC FREE-RESPONSE QUESTIONS

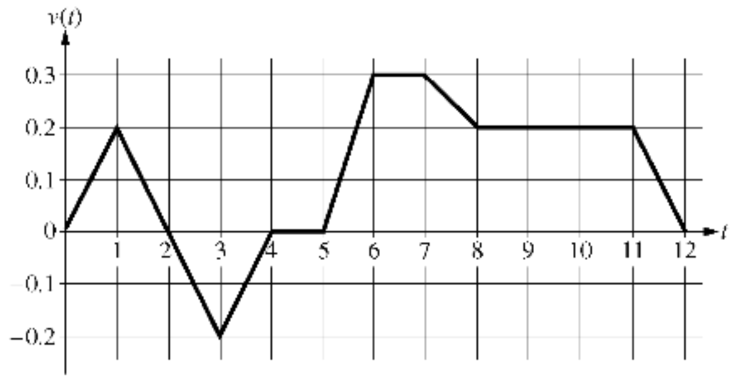
CALCULUS BC

SECTION II, Part A

Time—45 minutes

Number of problems—3

A graphing calculator is required for some problems or parts of problems.



1. Caren rides her bicycle along a straight road from home to school, starting at home at time $t = 0$ minutes and arriving at school at time $t = 12$ minutes. During the time interval $0 \le t \le 12$ minutes, her velocity $v(t)$, in miles per minute, is modeled by the piecewise-linear function whose graph is shown above.

   (a) Find the acceleration of Caren's bicycle at time $t = 7.5$ minutes. Indicate units of measure.

   (b) Using correct units, explain the meaning of $\int_0^{12} |v(t)|\, dt$ in terms of Caren's trip. Find the value of $\int_0^{12} |v(t)|\, dt$.

   (c) Shortly after leaving home, Caren realizes she left her calculus homework at home, and she returns to get it. At what time does she turn around to go back home? Give a reason for your answer.

   (d) Larry also rides his bicycle along a straight road from home to school in 12 minutes. His velocity is modeled by the function w given by $w(t) = \frac{\pi}{15}\sin\left(\frac{\pi}{12}t\right)$, where $w(t)$ is in miles per minute for $0 \le t \le 12$ minutes. Who lives closer to school: Caren or Larry? Show the work that leads to your answer.

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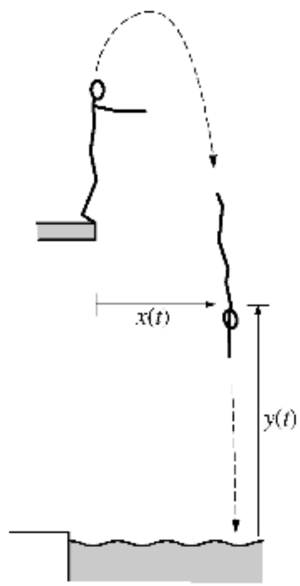
2. The rate at which people enter an auditorium for a rock concert is modeled by the function R given by $R(t) = 1380t^2 - 675t^3$ for $0 \le t \le 2$ hours; $R(t)$ is measured in people per hour. No one is in the auditorium at time $t = 0$, when the doors open. The doors close and the concert begins at time $t = 2$.

   (a) How many people are in the auditorium when the concert begins?

   (b) Find the time when the rate at which people enter the auditorium is a maximum. Justify your answer.

   (c) The total wait time for all the people in the auditorium is found by adding the time each person waits, starting at the time the person enters the auditorium and ending when the concert begins. The function w models the total wait time for all the people who enter the auditorium before time t. The derivative of w is given by $w'(t) = (2 - t)R(t)$. Find $w(2) - w(1)$, the total wait time for those who enter the auditorium after time $t = 1$.

   (d) On average, how long does a person wait in the auditorium for the concert to begin? Consider all people who enter the auditorium after the doors open, and use the model for total wait time from part (c).



Note: Figure not drawn to scale.

3. A diver leaps from the edge of a diving platform into a pool below. The figure above shows the initial position of the diver and her position at a later time. At time t seconds after she leaps, the horizontal distance from the front edge of the platform to the diver's shoulders is given by $x(t)$, and the vertical distance from the water surface to her shoulders is given by $y(t)$, where $x(t)$ and $y(t)$ are measured in meters. Suppose that the diver's shoulders are 11.4 meters above the water when she makes her leap and that

$$\frac{dx}{dt} = 0.8 \text{ and } \frac{dy}{dt} = 3.6 - 9.8t,$$

for $0 \le t \le A$, where A is the time that the diver's shoulders enter the water.

(a) Find the maximum vertical distance from the water surface to the diver's shoulders.

(b) Find A, the time that the diver's shoulders enter the water.

(c) Find the total distance traveled by the diver's shoulders from the time she leaps from the platform until the time her shoulders enter the water.

(d) Find the angle θ, $0 < \theta < \frac{\pi}{2}$, between the path of the diver and the water at the instant the diver's shoulders enter the water.

CALCULUS BC
SECTION II, Part B
Time—45 minutes
Number of problems—3

No calculator is allowed for these problems.

4. Consider the differential equation $\frac{dy}{dx} = 6x^2 - x^2y$. Let $y = f(x)$ be a particular solution to this differential equation with the initial condition $f(-1) = 2$.

   (a) Use Euler's method with two steps of equal size, starting at $x = -1$, to approximate $f(0)$. Show the work that leads to your answer.

   (b) At the point $(-1, 2)$, the value of $\frac{d^2y}{dx^2}$ is -12. Find the second-degree Taylor polynomial for f about $x = -1$.

   (c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(-1) = 2$.

---

x	2	3	5	8	13
$f(x)$	1	4	–2	3	6

5. Let f be a function that is twice differentiable for all real numbers. The table above gives values of f for selected points in the closed interval $2 \le x \le 13$.

   (a) Estimate $f'(4)$. Show the work that leads to your answer.

   (b) Evaluate $\int_2^{13} (3 - 5f'(x))\,dx$. Show the work that leads to your answer.

   (c) Use a left Riemann sum with subintervals indicated by the data in the table to approximate $\int_2^{13} f(x)\,dx$. Show the work that leads to your answer.

   (d) Suppose $f'(5) = 3$ and $f''(x) < 0$ for all x in the closed interval $5 \le x \le 8$. Use the line tangent to the graph of f at $x = 5$ to show that $f(7) \le 4$. Use the secant line for the graph of f on $5 \le x \le 8$ to show that $f(7) \ge \frac{4}{3}$.

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6. The Maclaurin series for e^x is $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!} + \cdots$. The continuous function f is defined by $f(x) = \frac{e^{(x-1)^2} - 1}{(x-1)^2}$ for $x \ne 1$ and $f(1) = 1$. The function f has derivatives of all orders at $x = 1$.

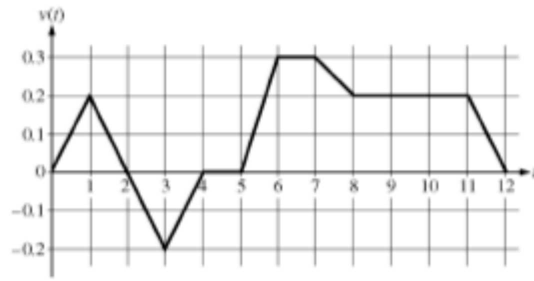
   (a) Write the first four nonzero terms and the general term of the Taylor series for $e^{(x-1)^2}$ about $x = 1$.

   (b) Use the Taylor series found in part (a) to write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

   (c) Use the ratio test to find the interval of convergence for the Taylor series found in part (b).

   (d) Use the Taylor series for f about $x = 1$ to determine whether the graph of f has any points of inflection.

Question 1



Caren rides her bicycle along a straight road from home to school, starting at home at time $t = 0$ minutes and arriving at school at time $t = 12$ minutes. During the time interval $0 \le t \le 12$ minutes, her velocity $v(t)$, in miles per minute, is modeled by the piecewise-linear function whose graph is shown above.

(a) Find the acceleration of Caren's bicycle at time $t = 7.5$ minutes. Indicate units of measure.

(b) Using correct units, explain the meaning of $\int_0^{12} |v(t)|\, dt$ in terms of Caren's trip. Find the value of $\int_0^{12} |v(t)|\, dt$.

(c) Shortly after leaving home, Caren realizes she left her calculus homework at home, and she returns to get it. At what time does she turn around to go back home? Give a reason for your answer.

(d) Larry also rides his bicycle along a straight road from home to school in 12 minutes. His velocity is modeled by the function w given by $w(t) = \frac{\pi}{15}\sin\left(\frac{\pi}{12}t\right)$, where $w(t)$ is in miles per minute for $0 \le t \le 12$ minutes. Who lives closer to school: Caren or Larry? Show the work that leads to your answer.

(a) $a(7.5) = v'(7.5) = \frac{v(8) - v(7)}{8 - 7} = -0.1$ miles/minute^2	2 : { 1 : answer 1 : units
(b) $\int_0^{12} \lvert v(t) \rvert\, dt$ is the total distance, in miles, that Caren rode during the 12 minutes from $t = 0$ to $t = 12$. $\int_0^{12} \lvert v(t) \rvert\, dt = \int_0^2 v(t)\, dt - \int_2^4 v(t)\, dt + \int_4^{12} v(t)\, dt$ $= 0.2 + 0.2 + 1.4 = 1.8$ miles	2 : { 1 : meaning of integral 1 : value of integral
(c) Caren turns around to go back home at time $t = 2$ minutes. This is the time at which her velocity changes from positive to negative.	2 : { 1 : answer 1 : reason
(d) $\int_0^{12} w(t)\, dt = 1.6$; Larry lives 1.6 miles from school. $\int_0^{12} v(t)\, dt = 1.4$; Caren lives 1.4 miles from school. Therefore, Caren lives closer to school.	3 : { 2 : Larry's distance from school 1 : integral 1 : value 1 : Caren's distance from school and conclusion

Question 2

The rate at which people enter an auditorium for a rock concert is modeled by the function R given by $R(t) = 1380t^2 - 675t^3$ for $0 \le t \le 2$ hours; $R(t)$ is measured in people per hour. No one is in the auditorium at time $t = 0$, when the doors open. The doors close and the concert begins at time $t = 2$.

(a) How many people are in the auditorium when the concert begins?

(b) Find the time when the rate at which people enter the auditorium is a maximum. Justify your answer.

(c) The total wait time for all the people in the auditorium is found by adding the time each person waits, starting at the time the person enters the auditorium and ending when the concert begins. The function w models the total wait time for all the people who enter the auditorium before time t. The derivative of w is given by $w'(t) = (2 - t)R(t)$. Find $w(2) - w(1)$, the total wait time for those who enter the auditorium after time $t = 1$.

(d) On average, how long does a person wait in the auditorium for the concert to begin? Consider all people who enter the auditorium after the doors open, and use the model for total wait time from part (c).

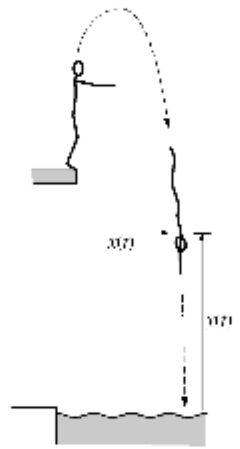
(a) $\int_0^2 R(t)\,dt = 980$ people	$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$
(b) $R'(t) = 0$ when $t = 0$ and $t = 1.36296$ The maximum rate may occur at 0, $a = 1.36296$, or 2. $R(0) = 0$ $R(a) = 854.527$ $R(2) = 120$ The maximum rate occurs when $t = 1.362$ or 1.363.	$3: \begin{cases} 1: \text{considers } R'(t) = 0 \\ 1: \text{interior critical point} \\ 1: \text{answer and justification} \end{cases}$
(c) $w(2) - w(1) = \int_1^2 w'(t)\,dt = \int_1^2 (2 - t)R(t)\,dt = 387.5$ The total wait time for those who enter the auditorium after time $t = 1$ is 387.5 hours.	$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$
(d) $\frac{1}{980}w(2) = \frac{1}{980}\int_0^2 (2 - t)R(t)\,dt = 0.77551$ On average, a person waits 0.775 or 0.776 hour.	$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$

Question 3

A diver leaps from the edge of a diving platform into a pool below. The figure above shows the initial position of the diver and her position at a later time. At time t seconds after she leaps, the horizontal distance from the front edge of the platform to the diver's shoulders is given by $x(t)$, and the vertical distance from the water surface to her shoulders is given by $y(t)$, where $x(t)$ and $y(t)$ are measured in meters. Suppose that the diver's shoulders are 11.4 meters above the water when she makes her leap and that

$$\frac{dx}{dt} = 0.8 \text{ and } \frac{dy}{dt} = 3.6 - 9.8t,$$

for $0 \le t \le A$, where A is the time that the diver's shoulders enter the water.



Note: Figure not drawn to scale.

(a) Find the maximum vertical distance from the water surface to the diver's shoulders.

(b) Find A, the time that the diver's shoulders enter the water.

(c) Find the total distance traveled by the diver's shoulders from the time she leaps from the platform until the time her shoulders enter the water.

(d) Find the angle θ, $0 < \theta < \frac{\pi}{2}$, between the path of the diver and the water at the instant the diver's shoulders enter the water.

(a) $\frac{dy}{dt} = 0$ only when $t = 0.36735$. Let $b = 0.36735$.
The maximum vertical distance from the water surface to the diver's shoulders is
$y(b) = 11.4 + \int_0^b \frac{dy}{dt}\,dt = 12.061$ meters.

Alternatively, $y(t) = 11.4 + 3.6t - 4.9t^2$, so $y(b) = 12.061$ meters.

3 : { 1 : considers $\frac{dy}{dt} = 0$; 1 : integral or $y(t)$; 1 : answer }

(b) $y(A) = 11.4 + \int_0^A \frac{dy}{dt}\,dt = 11.4 + 3.6A - 4.9A^2 = 0$ when
$A = 1.936$ seconds.

2 : { 1 : equation; 1 : answer }

(c) $\int_0^A \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = 12.946$ meters

2 : { 1 : integral; 1 : answer }

(d) At time A, $\frac{dy}{dx} = \left.\frac{dy/dt}{dx/dt}\right|_{t=A} = -19.21913$.

The angle between the path of the diver and the water is
$\tan^{-1}(19.21913) = 1.518$ or 1.519.

2 : { 1 : $\frac{dy}{dx}$ at time A; 1 : answer }

Question 4

Consider the differential equation $\frac{dy}{dx} = 6x^2 - x^2 y$. Let $y = f(x)$ be a particular solution to this differential equation with the initial condition $f(-1) = 2$.

(a) Use Euler's method with two steps of equal size, starting at $x = -1$, to approximate $f(0)$. Show the work that leads to your answer.

(b) At the point $(-1, 2)$, the value of $\frac{d^2y}{dx^2}$ is -12. Find the second-degree Taylor polynomial for f about $x = -1$.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(-1) = 2$.

(a) $f\left(-\frac{1}{2}\right) \approx f(-1) + \left(\left.\frac{dy}{dx}\right|_{(-1,2)}\right) \cdot \Delta x$

$= 2 + 4 \cdot \frac{1}{2} = 4$

$f(0) \approx f\left(-\frac{1}{2}\right) + \left(\left.\frac{dy}{dx}\right|_{\left(-\frac{1}{2},4\right)}\right) \cdot \Delta x$

$\approx 4 + \frac{1}{2} \cdot \frac{1}{2} = \frac{17}{4}$

$2: \begin{cases} 1: \text{Euler's method with two steps} \\ 1: \text{answer} \end{cases}$

(b) $P_2(x) = 2 + 4(x+1) - 6(x+1)^2$

1 : answer

(c) $\frac{dy}{dx} = x^2(6 - y)$

$\int \frac{1}{6-y}\,dy = \int x^2\,dx$

$-\ln|6-y| = \frac{1}{3}x^3 + C$

$-\ln 4 = -\frac{1}{3} + C$

$C = \frac{1}{3} - \ln 4$

$\ln|6-y| = -\frac{1}{3}x^3 - \left(\frac{1}{3} - \ln 4\right)$

$|6-y| = 4e^{-\frac{1}{3}(x^3+1)}$

$y = 6 - 4e^{-\frac{1}{3}(x^3+1)}$

$6: \begin{cases} 1: \text{separation of variables} \\ 2: \text{antiderivatives} \\ 1: \text{constant of integration} \\ 1: \text{uses initial condition} \\ 1: \text{solves for } y \end{cases}$

Note: max $3/6$ [1-2-0-0-0] if no constant of integration

Note: $0/6$ if no separation of variables

Question 5

x	2	3	5	8	13
$f(x)$	1	4	–2	3	6

Let f be a function that is twice differentiable for all real numbers. The table above gives values of f for selected points in the closed interval $2 \le x \le 13$.

(a) Estimate $f'(4)$. Show the work that leads to your answer.

(b) Evaluate $\int_2^{13} (3 - 5f'(x))\,dx$. Show the work that leads to your answer.

(c) Use a left Riemann sum with subintervals indicated by the data in the table to approximate $\int_2^{13} f(x)\,dx$. Show the work that leads to your answer.

(d) Suppose $f'(5) = 3$ and $f''(x) < 0$ for all x in the closed interval $5 \le x \le 8$. Use the line tangent to the graph of f at $x = 5$ to show that $f(7) \le 4$. Use the secant line for the graph of f on $5 \le x \le 8$ to show that $f(7) \ge \frac{4}{3}$.

Solution	Scoring
(a) $f'(4) \approx \dfrac{f(5) - f(3)}{5 - 3} = -3$	1 : answer
(b) $\int_2^{13} (3 - 5f'(x))\,dx = \int_2^{13} 3\,dx - 5\int_2^{13} f'(x)\,dx$ $= 3(13 - 2) - 5(f(13) - f(2)) = 8$	2 : { 1 : uses Fundamental Theorem of Calculus; 1 : answer }
(c) $\int_2^{13} f(x)\,dx \approx f(2)(3-2) + f(3)(5-3) + f(5)(8-5) + f(8)(13-8) = 18$	2 : { 1 : left Riemann sum; 1 : answer }
(d) An equation for the tangent line is $y = -2 + 3(x - 5)$. Since $f''(x) < 0$ for all x in the interval $5 \le x \le 8$, the line tangent to the graph of $y = f(x)$ at $x = 5$ lies above the graph for all x in the interval $5 < x \le 8$. Therefore, $f(7) \le -2 + 3 \cdot 2 = 4$. An equation for the secant line is $y = -2 + \frac{5}{3}(x - 5)$. Since $f''(x) < 0$ for all x in the interval $5 \le x \le 8$, the secant line connecting $(5, f(5))$ and $(8, f(8))$ lies below the graph of $y = f(x)$ for all x in the interval $5 < x < 8$. Therefore, $f(7) \ge -2 + \frac{5}{3} \cdot 2 = \frac{4}{3}$.	4 : { 1 : tangent line; 1 : shows $f(7) \le 4$; 1 : secant line; 1 : shows $f(7) \ge \frac{4}{3}$ }

Question 6

The Maclaurin series for e^x is $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!} + \cdots$. The continuous function f is defined by $f(x) = \frac{e^{(x-1)^2} - 1}{(x-1)^2}$ for $x \neq 1$ and $f(1) = 1$. The function f has derivatives of all orders at $x = 1$.

(a) Write the first four nonzero terms and the general term of the Taylor series for $e^{(x-1)^2}$ about $x = 1$.

(b) Use the Taylor series found in part (a) to write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

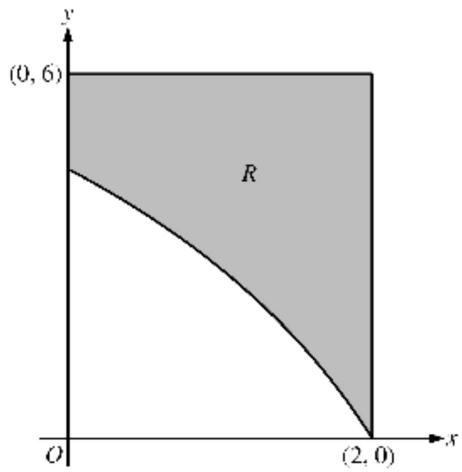
(c) Use the ratio test to find the interval of convergence for the Taylor series found in part (b).

(d) Use the Taylor series for f about $x = 1$ to determine whether the graph of f has any points of inflection.

Solution	Scoring
(a) $1 + (x-1)^2 + \frac{(x-1)^4}{2} + \frac{(x-1)^6}{6} + \cdots + \frac{(x-1)^{2n}}{n!} + \cdots$	2 : { 1 : first four terms; 1 : general term
(b) $1 + \frac{(x-1)^2}{2} + \frac{(x-1)^4}{6} + \frac{(x-1)^6}{24} + \cdots + \frac{(x-1)^{2n}}{(n+1)!} + \cdots$	2 : { 1 : first four terms; 1 : general term
(c) $\lim_{n\to\infty} \left\| \frac{\frac{(x-1)^{2n+2}}{(n+2)!}}{\frac{(x-1)^{2n}}{(n+1)!}} \right\| = \lim_{n\to\infty} \frac{(n+1)!}{(n+2)!}(x-1)^2 = \lim_{n\to\infty} \frac{(x-1)^2}{n+2} = 0$ Therefore, the interval of convergence is $(-\infty, \infty)$.	3 : { 1 : sets up ratio; 1 : computes limit of ratio; 1 : answer
(d) $f''(x) = 1 + \frac{4\cdot 3}{6}(x-1)^2 + \frac{6\cdot 5}{24}(x-1)^4 + \cdots + \frac{2n(2n-1)}{(n+1)!}(x-1)^{2n-2} + \cdots$ Since every term of this series is nonnegative, $f''(x) \geq 0$ for all x. Therefore, the graph of f has no points of inflection.	2 : { 1 : $f''(x)$; 1 : answer

CALCULUS BC
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.



1. In the figure above, R is the shaded region in the first quadrant bounded by the graph of $y = 4\ln(3 - x)$, the horizontal line $y = 6$, and the vertical line $x = 2$.

   (a) Find the area of R.

   (b) Find the volume of the solid generated when R is revolved about the horizontal line $y = 8$.

   (c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of the solid.

---

2. The velocity vector of a particle moving in the xy-plane has components given by

$$\frac{dx}{dt} = 14\cos\left(t^2\right)\sin\left(e^t\right) \text{ and } \frac{dy}{dt} = 1 + 2\sin\left(t^2\right), \text{ for } 0 \le t \le 1.5.$$

   At time $t = 0$, the position of the particle is $(-2, 3)$.

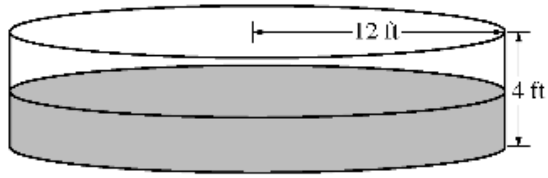
   (a) For $0 < t < 1.5$, find all values of t at which the line tangent to the path of the particle is vertical.

   (b) Write an equation for the line tangent to the path of the particle at $t = 1$.

   (c) Find the speed of the particle at $t = 1$.

   (d) Find the acceleration vector of the particle at $t = 1$.

t	0	2	4	6	8	10	12
$P(t)$	0	46	53	57	60	62	63



3. The figure above shows an aboveground swimming pool in the shape of a cylinder with a radius of 12 feet and a height of 4 feet. The pool contains 1000 cubic feet of water at time $t = 0$. During the time interval $0 \le t \le 12$ hours, water is pumped into the pool at the rate $P(t)$ cubic feet per hour. The table above gives values of $P(t)$ for selected values of t. During the same time interval, water is leaking from the pool at the rate $R(t)$ cubic feet per hour, where $R(t) = 25e^{-0.05t}$. (Note: The volume V of a cylinder with radius r and height h is given by $V = \pi r^2 h$.)

(a) Use a midpoint Riemann sum with three subintervals of equal length to approximate the total amount of water that was pumped into the pool during the time interval $0 \le t \le 12$ hours. Show the computations that lead to your answer.

(b) Calculate the total amount of water that leaked out of the pool during the time interval $0 \le t \le 12$ hours.

(c) Use the results from parts (a) and (b) to approximate the volume of water in the pool at time $t = 12$ hours. Round your answer to the nearest cubic foot.

(d) Find the rate at which the volume of water in the pool is increasing at time $t = 8$ hours. How fast is the water level in the pool rising at $t = 8$ hours? Indicate units of measure in both answers.

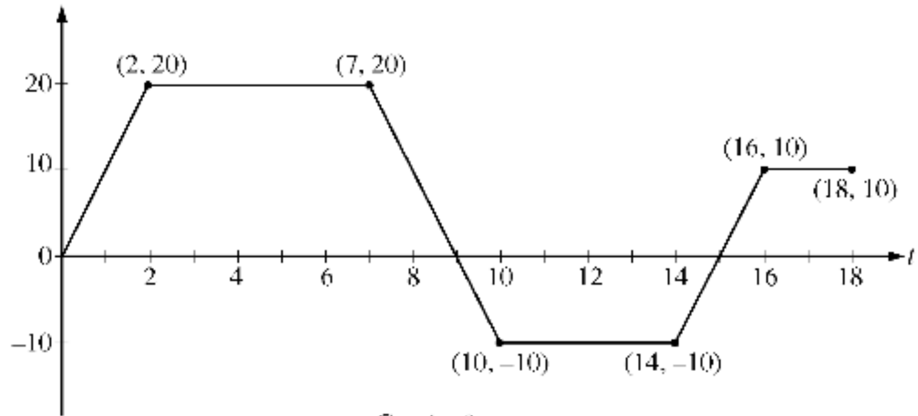
CALCULUS BC

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.



Graph of v

4. A squirrel starts at building A at time $t = 0$ and travels along a straight, horizontal wire connected to building B. For $0 \le t \le 18$, the squirrel's velocity is modeled by the piecewise-linear function defined by the graph above.

   (a) At what times in the interval $0 < t < 18$, if any, does the squirrel change direction? Give a reason for your answer.

   (b) At what time in the interval $0 \le t \le 18$ is the squirrel farthest from building A ? How far from building A is the squirrel at that time?

   (c) Find the total distance the squirrel travels during the time interval $0 \le t \le 18$.

   (d) Write expressions for the squirrel's acceleration $a(t)$, velocity $v(t)$, and distance $x(t)$ from building A that are valid for the time interval $7 < t < 10$.

---

5. Let f and g be the functions defined by $f(x) = \frac{1}{x}$ and $g(x) = \frac{4x}{1 + 4x^2}$, for all $x > 0$.

   (a) Find the absolute maximum value of g on the open interval $(0, \infty)$ if the maximum exists. Find the absolute minimum value of g on the open interval $(0, \infty)$ if the minimum exists. Justify your answers.

   (b) Find the area of the unbounded region in the first quadrant to the right of the vertical line $x = 1$, below the graph of f, and above the graph of g.

---

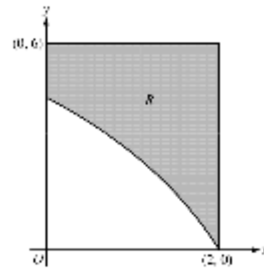
6. The Maclaurin series for the function f is given by $f(x) = \sum_{n=2}^{\infty} \frac{(-1)^n (2x)^n}{n - 1}$ on its interval of convergence.

   (a) Find the interval of convergence for the Maclaurin series of f. Justify your answer.

   (b) Show that $y = f(x)$ is a solution to the differential equation $xy' - y = \frac{4x^2}{1 + 2x}$ for $|x| < R$, where R is the radius of convergence from part (a).

Question 1

In the figure above, R is the shaded region in the first quadrant bounded by the graph of $y = 4\ln(3 - x)$, the horizontal line $y = 6$, and the vertical line $x = 2$.



(a) Find the area of R.

(b) Find the volume of the solid generated when R is revolved about the horizontal line $y = 8$.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of the solid.

---

1 : Correct limits in an integral in (a), (b), or (c)

(a) $\int_0^2 (6 - 4\ln(3 - x))\,dx = 6.816$ or 6.817

2 : { 1 : integrand; 1 : answer }

(b) $\pi\int_0^2 \left((8 - 4\ln(3 - x))^2 - (8 - 6)^2\right) dx$
$= 168.179$ or 168.180

3 : { 2 : integrand; 1 : answer }

(c) $\int_0^2 (6 - 4\ln(3 - x))^2\,dx = 26.266$ or 26.267

3 : { 2 : integrand; 1 : answer }

Question 2

The velocity vector of a particle moving in the plane has components given by

$$\frac{dx}{dt} = 14\cos(t^2)\sin(e^t) \text{ and } \frac{dy}{dt} = 1 + 2\sin(t^2), \text{ for } 0 \le t \le 1.5.$$

At time $t = 0$, the position of the particle is $(-2, 3)$.

(a) For $0 < t < 1.5$, find all values of t at which the line tangent to the path of the particle is vertical.

(b) Write an equation for the line tangent to the path of the particle at $t = 1$.

(c) Find the speed of the particle at $t = 1$.

(d) Find the acceleration vector of the particle at $t = 1$.

(a) The tangent line is vertical when $x'(t) = 0$ and $y'(t) \ne 0$. On $0 < t < 1.5$, this happens at $t = 1.253$ and $t = 1.144$ or 1.145.	$2: \begin{cases} 1: \text{sets } \frac{dx}{dt} = 0 \\ 1: \text{answer} \end{cases}$
(b) $\left.\frac{dy}{dx}\right\vert_{t=1} = \frac{y'(1)}{x'(1)} = 0.863447$ $x(1) = -2 + \int_0^1 x'(t)\,dt = 9.314695$ $y(1) = 3 + \int_0^1 y'(t)\,dt = 4.620537$ The line tangent to the path of the particle at $t = 1$ has equation $y = 4.621 + 0.863(x - 9.315)$.	$4: \begin{cases} 1: \left.\frac{dy}{dx}\right\vert_{t=1} \\ 1: x(1) \\ 1: y(1) \\ 1: \text{equation} \end{cases}$
(c) Speed $= \sqrt{(x'(1))^2 + (y'(1))^2} = 4.105$	1 : answer
(d) Acceleration vector: $\langle x''(1), y''(1)\rangle = \langle -28.425, 2.161\rangle$	$2: \begin{cases} 1: x''(1) \\ 1: y''(1) \end{cases}$

Question 3

t	0	2	4	6	8	10	12
$P(t)$	0	46	53	57	60	62	63

The figure above shows an aboveground swimming pool in the shape of a cylinder with a radius of 12 feet and a height of 4 feet. The pool contains 1000 cubic feet of water at time $t = 0$. During the time interval $0 \le t \le 12$ hours, water is pumped into the pool at the rate $P(t)$ cubic feet per hour. The table above gives values of $P(t)$ for selected values of t. During the same time interval, water is leaking from the pool at the rate $R(t)$ cubic feet per hour, where $R(t) = 25e^{-0.05t}$. (Note: The volume V of a cylinder with radius r and height h is given by $V = \pi r^2 h$.)

(a) Use a midpoint Riemann sum with three subintervals of equal length to approximate the total amount of water that was pumped into the pool during the time interval $0 \le t \le 12$ hours. Show the computations that lead to your answer.

(b) Calculate the total amount of water that leaked out of the pool during the time interval $0 \le t \le 12$ hours.

(c) Use the results from parts (a) and (b) to approximate the volume of water in the pool at time $t = 12$ hours. Round your answer to the nearest cubic foot.

(d) Find the rate at which the volume of water in the pool is increasing at time $t = 8$ hours. How fast is the water level in the pool rising at $t = 8$ hours? Indicate units of measure in both answers.

(a) $\int_0^{12} P(t)\,dt \approx 46 \cdot 4 + 57 \cdot 4 + 62 \cdot 4 = 660 \text{ ft}^3$

$2: \begin{cases} 1: \text{midpoint sum} \\ 1: \text{answer} \end{cases}$

(b) $\int_0^{12} R(t)\,dt = 225.594 \text{ ft}^3$

$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$

(c) $1000 + \int_0^{12} P(t)\,dt - \int_0^{12} R(t)\,dt = 1434.406$

At time $t = 12$ hours, the volume of water in the pool is approximately 1434 ft^3.

1 : answer

(d) $V'(t) = P(t) - R(t)$

$V'(8) = P(8) - R(8) = 60 - 25e^{-0.4} = 43.241 \text{ or } 43.242 \text{ ft}^3/\text{hr}$

$V = \pi(12)^2 h$

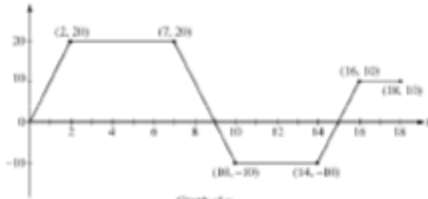
$\frac{dV}{dt} = 144\pi \frac{dh}{dt}$

$\left.\frac{dh}{dt}\right|_{t=8} = \frac{1}{144\pi} \cdot \left.\frac{dV}{dt}\right|_{t=8} = 0.095 \text{ or } 0.096 \text{ ft/hr}$

$4: \begin{cases} 1: V'(8) \\ 1: \text{equation relating } \frac{dV}{dt} \text{ and } \frac{dh}{dt} \\ 1: \left.\frac{dh}{dt}\right|_{t=8} \\ 1: \text{units of ft}^3/\text{hr and ft/hr} \end{cases}$

Question 4

A squirrel starts at building A at time $t = 0$ and travels along a straight wire connected to building B. For $0 \le t \le 18$, the squirrel's velocity is modeled by the piecewise-linear function defined by the graph above.



Graph of v

(a) At what times in the interval $0 < t < 18$, if any, does the squirrel change direction? Give a reason for your answer.

(b) At what time in the interval $0 \le t \le 18$ is the squirrel farthest from building A? How far from building A is the squirrel at this time?

(c) Find the total distance the squirrel travels during the time interval $0 \le t \le 18$.

(d) Write expressions for the squirrel's acceleration $a(t)$, velocity $v(t)$, and distance $x(t)$ from building A that are valid for the time interval $7 < t < 10$.

(a) The squirrel changes direction whenever its velocity changes sign. This occurs at $t = 9$ and $t = 15$.

$2 : \begin{cases} 1 : t\text{-values} \\ 1 : \text{explanation} \end{cases}$

(b) Velocity is 0 at $t = 0$, $t = 9$, and $t = 15$.

t	position at time t
0	0
9	$\frac{9+5}{2} \cdot 20 = 140$
15	$140 - \frac{6+4}{2} \cdot 10 = 90$
18	$90 + \frac{3+2}{2} \cdot 10 = 115$

The squirrel is farthest from building A at time $t = 9$; its greatest distance from the building is 140.

$2 : \begin{cases} 1 : \text{identifies candidates} \\ 1 : \text{answers} \end{cases}$

(c) The total distance traveled is $\int_0^{18} |v(t)|\, dt = 140 + 50 + 25 = 215$.

$1 : \text{answer}$

(d) For $7 < t < 10$, $a(t) = \frac{20 - (-10)}{7 - 10} = -10$

$v(t) = 20 - 10(t - 7) = -10t + 90$

$x(7) = \frac{7+5}{2} \cdot 20 = 120$

$x(t) = x(7) + \int_7^t (-10u + 90)\, du$

$= 120 + \left(-5u^2 + 90u\right)\Big|_{u=7}^{u=t}$

$= -5t^2 + 90t - 265$

$4 : \begin{cases} 1 : a(t) \\ 1 : v(t) \\ 2 : x(t) \end{cases}$

Question 5

Let f and g be the functions defined by $f(x) = \frac{1}{x}$ and $g(x) = \frac{4x}{1+4x^2}$, for all $x > 0$.

(a) Find the absolute maximum value of g on the open interval $(0, \infty)$ if the maximum exists. Find the absolute minimum value of g on the open interval $(0, \infty)$ if the minimum exists. Justify your answers.

(b) Find the area of the unbounded region in the first quadrant to the right of the vertical line $x = 1$, below the graph of f, and above the graph of g.

(a) $g'(x) = \frac{4(1+4x^2) - 4x(8x)}{(1+4x^2)^2} = \frac{4(1-4x^2)}{(1+4x^2)^2}$

For $x > 0$, $g'(x) = 0$ for $x = \frac{1}{2}$.

$g'(x) > 0$ for $0 < x < \frac{1}{2}$

$g'(x) < 0$ for $x > \frac{1}{2}$

$g\left(\frac{1}{2}\right) = 1$

Therefore g has a maximum value of 1 at $x = \frac{1}{2}$, and g has no minimum value on the open interval $(0, \infty)$.

$5 : \begin{cases} 2 : g'(x) \\ 1 : \text{critical point} \\ 1 : \text{answers} \\ 1 : \text{justification} \end{cases}$

(b) $\int_1^{\infty} (f(x) - g(x))\,dx = \lim_{b \to \infty} \int_1^{b} (f(x) - g(x))\,dx$

$= \lim_{b \to \infty} \left(\ln(x) - \frac{1}{2}\ln(1+4x^2) \right) \Big|_{x=1}^{x=b}$

$= \lim_{b \to \infty} \left(\ln(b) - \frac{1}{2}\ln(1+4b^2) + \frac{1}{2}\ln(5) \right)$

$= \lim_{b \to \infty} \ln\left(\frac{b\sqrt{5}}{\sqrt{1+4b^2}} \right)$

$= \lim_{b \to \infty} \ln\left(\frac{\sqrt{5b^2}}{\sqrt{1+4b^2}} \right)$

$= \frac{1}{2} \lim_{b \to \infty} \ln\left(\frac{5b^2}{1+4b^2} \right)$

$= \frac{1}{2} \ln \frac{5}{4}$

$4 : \begin{cases} 1 : \text{integral} \\ 2 : \text{antidifferentiation} \\ 1 : \text{answer} \end{cases}$

Question 6

The Maclaurin series for the function f is given by $f(x) = \sum_{n=2}^{\infty} \frac{(-1)^n (2x)^n}{n-1}$ on its interval of convergence.

(a) Find the interval of convergence for the Maclaurin series of f. Justify your answer.

(b) Show that $y = f(x)$ is a solution to the differential equation $xy' - y = \frac{4x^2}{1+2x}$ for $|x| < R$, where R is the radius of convergence from part (a).

(a) $$\lim_{n\to\infty} \left| \frac{\frac{(2x)^{n+1}}{(n+1)-1}}{\frac{(2x)^n}{n-1}} \right| = \lim_{n\to\infty} \left| 2x \cdot \frac{n-1}{n} \right| = \lim_{n\to\infty} \left| 2x \cdot \frac{n-1}{n} \right| = |2x|$$

$|2x| < 1$ for $|x| < \frac{1}{2}$

Therefore the radius of convergence is $\frac{1}{2}$.

When $x = -\frac{1}{2}$, the series is $\sum_{n=2}^{\infty} \frac{(-1)^n (-1)^n}{n-1} = \sum_{n=2}^{\infty} \frac{1}{n-1}$.

This is the harmonic series, which diverges.

When $x = \frac{1}{2}$, the series is $\sum_{n=2}^{\infty} \frac{(-1)^n 1^n}{n-1} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n-1}$.

This is the alternating harmonic series, which converges.

The interval of convergence for the Maclaurin series of f is $\left(-\frac{1}{2}, \frac{1}{2}\right]$.

5 :
- 1 : sets up ratio
- 1 : limit evaluation
- 1 : radius of convergence
- 1 : considers both endpoints
- 1 : analysis and interval of convergence

(b) $$y = \frac{(2x)^2}{1} - \frac{(2x)^3}{2} + \frac{(2x)^4}{3} - \cdots + \frac{(-1)^n (2x)^n}{n-1} + \cdots$$

$$= 4x^2 - 4x^3 + \frac{16}{3}x^4 - \cdots + \frac{(-1)^n (2x)^n}{n-1} + \cdots$$

$$y' = 8x - 12x^2 + \frac{64}{3}x^3 - \cdots + \frac{(-1)^n n(2x)^{n-1} \cdot 2}{n-1} + \cdots$$

$$xy' = 8x^2 - 12x^3 + \frac{64}{3}x^4 - \cdots + \frac{(-1)^n n(2x)^n}{n-1} + \cdots$$

$$xy' - y = 4x^2 - 8x^3 + 16x^4 - \cdots + (-1)^n (2x)^n + \cdots$$

$$= 4x^2 \left(1 - 2x + 4x^2 - \cdots + (-1)^n (2x)^{n-2} + \cdots\right)$$

The series $1 - 2x + 4x^2 - \cdots + (-1)^n (2x)^{n-2} + \cdots = \sum_{n=0}^{\infty} (-2x)^n$ is a geometric series that converges to $\frac{1}{1+2x}$ for $|x| < \frac{1}{2}$. Therefore $xy' - y = 4x^2 \cdot \frac{1}{1+2x}$ for $|x| < \frac{1}{2}$.

4 :
- 1 : series for y'
- 1 : series for xy'
- 1 : series for $xy' - y$
- 1 : analysis with geometric series

CALCULUS BC
SECTION II, Part A
Time—45 minutes
Number of problems—3

A graphing calculator is required for some problems or parts of problems.

1. There is no snow on Janet's driveway when snow begins to fall at midnight. From midnight to 9 A.M., snow accumulates on the driveway at a rate modeled by $f(t) = 7te^{\cos t}$ cubic feet per hour, where t is measured in hours since midnight. Janet starts removing snow at 6 A.M. $(t = 6)$. The rate $g(t)$, in cubic feet per hour, at which Janet removes snow from the driveway at time t hours after midnight is modeled by

$$g(t) = \begin{cases} 0 & \text{for } 0 \le t < 6 \\ 125 & \text{for } 6 \le t < 7 \\ 108 & \text{for } 7 \le t \le 9. \end{cases}$$

(a) How many cubic feet of snow have accumulated on the driveway by 6 A.M.?

(b) Find the rate of change of the volume of snow on the driveway at 8 A.M.

(c) Let $h(t)$ represent the total amount of snow, in cubic feet, that Janet has removed from the driveway at time t hours after midnight. Express h as a piecewise-defined function with domain $0 \le t \le 9$.

(d) How many cubic feet of snow are on the driveway at 9 A.M.?

---

t (hours)	0	2	5	7	8
$E(t)$ (hundreds of entries)	0	4	13	21	23

2. A zoo sponsored a one-day contest to name a new baby elephant. Zoo visitors deposited entries in a special box between noon $(t = 0)$ and 8 P.M. $(t = 8)$. The number of entries in the box t hours after noon is modeled by a differentiable function E for $0 \le t \le 8$. Values of $E(t)$, in hundreds of entries, at various times t are shown in the table above.

(a) Use the data in the table to approximate the rate, in hundreds of entries per hour, at which entries were being deposited at time $t = 6$. Show the computations that lead to your answer.

(b) Use a trapezoidal sum with the four subintervals given by the table to approximate the value of $\frac{1}{8}\int_0^8 E(t)\,dt$. Using correct units, explain the meaning of $\frac{1}{8}\int_0^8 E(t)\,dt$ in terms of the number of entries.

(c) At 8 P.M., volunteers began to process the entries. They processed the entries at a rate modeled by the function P, where $P(t) = t^3 - 30t^2 + 298t - 976$ hundreds of entries per hour for $8 \le t \le 12$. According to the model, how many entries had not yet been processed by midnight $(t = 12)$?

(d) According to the model from part (c), at what time were the entries being processed most quickly? Justify your answer.

---

3. A particle is moving along a curve so that its position at time t is $(x(t), y(t))$, where $x(t) = t^2 - 4t + 8$ and $y(t)$ is not explicitly given. Both x and y are measured in meters, and t is measured in seconds. It is known that $\frac{dy}{dt} = te^{t-3} - 1$.

(a) Find the speed of the particle at time $t = 3$ seconds.

(b) Find the total distance traveled by the particle for $0 \le t \le 4$ seconds.

(c) Find the time t, $0 \le t \le 4$, when the line tangent to the path of the particle is horizontal. Is the direction of motion of the particle toward the left or toward the right at that time? Give a reason for your answer.

(d) There is a point with x-coordinate 5 through which the particle passes twice. Find each of the following.

(i) The two values of t when that occurs

(ii) The slopes of the lines tangent to the particle's path at that point

(iii) The y-coordinate of that point, given $y(2) = 3 + \frac{1}{e}$

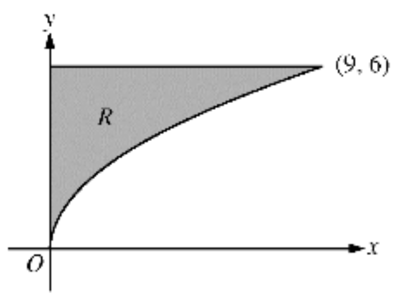
CALCULUS BC

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.



4. Let R be the region in the first quadrant bounded by the graph of $y = 2\sqrt{x}$, the horizontal line $y = 6$, and the y-axis, as shown in the figure above.

(a) Find the area of R.

(b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 7$.

(c) Region R is the base of a solid. For each y, where $0 \le y \le 6$, the cross section of the solid taken perpendicular to the y-axis is a rectangle whose height is 3 times the length of its base in region R. Write, but do not evaluate, an integral expression that gives the volume of the solid.

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5. Consider the differential equation $\frac{dy}{dx} = 1 - y$. Let $y = f(x)$ be the particular solution to this differential equation with the initial condition $f(1) = 0$. For this particular solution, $f(x) < 1$ for all values of x.

(a) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(0)$. Show the work that leads to your answer.

(b) Find $\lim_{x \to 1} \frac{f(x)}{x^3 - 1}$. Show the work that leads to your answer.

(c) Find the particular solution $y = f(x)$ to the differential equation $\frac{dy}{dx} = 1 - y$ with the initial condition $f(1) = 0$.

$$f(x) = \begin{cases} \frac{\cos x - 1}{x^2} & \text{for } x \neq 0 \\ -\frac{1}{2} & \text{for } x = 0 \end{cases}$$

6. The function f, defined above, has derivatives of all orders. Let g be the function defined by $g(x) = 1 + \int_0^x f(t)\, dt$.

(a) Write the first three nonzero terms and the general term of the Taylor series for $\cos x$ about $x = 0$. Use this series to write the first three nonzero terms and the general term of the Taylor series for f about $x = 0$.

(b) Use the Taylor series for f about $x = 0$ found in part (a) to determine whether f has a relative maximum, relative minimum, or neither at $x = 0$. Give a reason for your answer.

(c) Write the fifth-degree Taylor polynomial for g about $x = 0$.

(d) The Taylor series for g about $x = 0$, evaluated at $x = 1$, is an alternating series with individual terms that decrease in absolute value to 0. Use the third-degree Taylor polynomial for g about $x = 0$ to estimate the value of $g(1)$. Explain why this estimate differs from the actual value of $g(1)$ by less than $\frac{1}{6!}$.

Question 1

There is no snow on Janet's driveway when snow begins to fall at midnight. From midnight to 9 A.M., snow accumulates on the driveway at a rate modeled by $f(t) = 7te^{\cos t}$ cubic feet per hour, where t is measured in hours since midnight. Janet starts removing snow at 6 A.M. $(t = 6)$. The rate $g(t)$, in cubic feet per hour, at which Janet removes snow from the driveway at time t hours after midnight is modeled by

$$g(t) = \begin{cases} 0 & \text{for } 0 \le t < 6 \\ 125 & \text{for } 6 \le t < 7 \\ 108 & \text{for } 7 \le t \le 9. \end{cases}$$

(a) How many cubic feet of snow have accumulated on the driveway by 6 A.M.?

(b) Find the rate of change of the volume of snow on the driveway at 8 A.M.

(c) Let $h(t)$ represent the total amount of snow, in cubic feet, that Janet has removed from the driveway at time t hours after midnight. Express h as a piecewise-defined function with domain $0 \le t \le 9$.

(d) How many cubic feet of snow are on the driveway at 9 A.M.?

(a) $\int_0^6 f(t)\,dt = 142.274$ or 142.275 cubic feet

$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$

(b) Rate of change is $f(8) - g(8) = -59.582$ or -59.583 cubic feet per hour.

$1: \text{answer}$

(c) $h(0) = 0$

For $0 < t \le 6$, $h(t) = h(0) + \int_0^t g(s)\,ds = 0 + \int_0^t 0\,ds = 0.$

For $6 < t \le 7$, $h(t) = h(6) + \int_6^t g(s)\,ds = 0 + \int_6^t 125\,ds = 125(t-6).$

For $7 < t \le 9$, $h(t) = h(7) + \int_7^t g(s)\,ds = 125 + \int_7^t 108\,ds = 125 + 108(t-7).$

Thus, $h(t) = \begin{cases} 0 & \text{for } 0 \le t \le 6 \\ 125(t-6) & \text{for } 6 < t \le 7 \\ 125 + 108(t-7) & \text{for } 7 < t \le 9 \end{cases}$

$3: \begin{cases} 1: h(t) \text{ for } 0 \le t \le 6 \\ 1: h(t) \text{ for } 6 < t \le 7 \\ 1: h(t) \text{ for } 7 < t \le 9 \end{cases}$

(d) Amount of snow is $\int_0^9 f(t)\,dt - h(9) = 26.334$ or 26.335 cubic feet.

$3: \begin{cases} 1: \text{integral} \\ 1: h(9) \\ 1: \text{answer} \end{cases}$

Question 2

t (hours)	0	2	5	7	8
$E(t)$ (hundreds of entries)	0	4	13	21	23

A zoo sponsored a one-day contest to name a new baby elephant. Zoo visitors deposited entries in a special box between noon $(t = 0)$ and 8 P.M. $(t = 8)$. The number of entries in the box t hours after noon is modeled by a differentiable function E for $0 \le t \le 8$. Values of $E(t)$, in hundreds of entries, at various times t are shown in the table above.

(a) Use the data in the table to approximate the rate, in hundreds of entries per hour, at which entries were being deposited at time $t = 6$. Show the computations that lead to your answer.

(b) Use a trapezoidal sum with the four subintervals given by the table to approximate the value of $\frac{1}{8}\int_0^8 E(t)\,dt$. Using correct units, explain the meaning of $\frac{1}{8}\int_0^8 E(t)\,dt$ in terms of the number of entries.

(c) At 8 P.M., volunteers began to process the entries. They processed the entries at a rate modeled by the function P, where $P(t) = t^3 - 30t^2 + 298t - 976$ hundreds of entries per hour for $8 \le t \le 12$. According to the model, how many entries had not yet been processed by midnight $(t = 12)$?

(d) According to the model from part (c), at what time were the entries being processed most quickly? Justify your answer.

(a) $E'(6) \approx \dfrac{E(7) - E(5)}{7 - 5} = 4$ hundred entries per hour

1 : answer

(b) $\frac{1}{8}\int_0^8 E(t)\,dt \approx$

$\frac{1}{8}\left(2\cdot\frac{E(0)+E(2)}{2} + 3\cdot\frac{E(2)+E(5)}{2} + 2\cdot\frac{E(5)+E(7)}{2} + 1\cdot\frac{E(7)+E(8)}{2}\right)$

$= 10.687$ or 10.688

$\frac{1}{8}\int_0^8 E(t)\,dt$ is the average number of hundreds of entries in the box between noon and 8 P.M.

3 : { 1 : trapezoidal sum; 1 : approximation; 1 : meaning }

(c) $23 - \int_8^{12} P(t)\,dt = 23 - 16 = 7$ hundred entries

2 : { 1 : integral; 1 : answer }

(d) $P'(t) = 0$ when $t = 9.183503$ and $t = 10.816497$.

t	$P(t)$
8	0
9.183503	5.088662
10.816497	2.911338
12	8

Entries are being processed most quickly at time $t = 12$.

3 : { 1 : considers $P'(t) = 0$; 1 : identifies candidates; 1 : answer with justification }

Question 3

A particle is moving along a curve so that its position at time t is $(x(t), y(t))$, where $x(t) = t^2 - 4t + 8$ and $y(t)$ is not explicitly given. Both x and y are measured in meters, and t is measured in seconds. It is known that $\frac{dy}{dt} = te^{t-3} - 1$.

(a) Find the speed of the particle at time $t = 3$ seconds.

(b) Find the total distance traveled by the particle for $0 \le t \le 4$ seconds.

(c) Find the time t, $0 \le t \le 4$, when the line tangent to the path of the particle is horizontal. Is the direction of motion of the particle toward the left or toward the right at that time? Give a reason for your answer.

(d) There is a point with x-coordinate 5 through which the particle passes twice. Find each of the following.

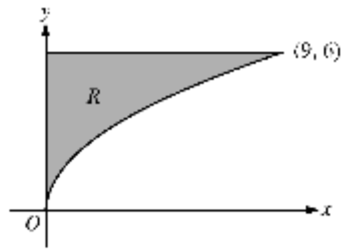
(i) The two values of t when that occurs

(ii) The slopes of the lines tangent to the particle's path at that point

(iii) The y-coordinate of that point, given $y(2) = 3 + \frac{1}{e}$

(a) Speed $= \sqrt{(x'(3))^2 + (y'(3))^2} = 2.828$ meters per second	1 : answer
(b) $x'(t) = 2t - 4$ Distance $= \int_0^4 \sqrt{(2t-4)^2 + \left(te^{t-3} - 1\right)^2}\, dt = 11.587$ or 11.588 meters	$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$
(c) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = 0$ when $te^{t-3} - 1 = 0$ and $2t - 4 \ne 0$ This occurs at $t = 2.20794$. Since $x'(2.20794) > 0$, the particle is moving toward the right at time $t = 2.207$ or 2.208.	$3 : \begin{cases} 1 : \text{considers } \frac{dy}{dx} = 0 \\ 1 : t = 2.207 \text{ or } 2.208 \\ 1 : \text{direction of motion with reason} \end{cases}$
(d) $x(t) = 5$ at $t = 1$ and $t = 3$ At time $t = 1$, the slope is $\left.\frac{dy}{dx}\right\|_{t=1} = \left.\frac{dy/dt}{dx/dt}\right\|_{t=1} = 0.432$. At time $t = 3$, the slope is $\left.\frac{dy}{dx}\right\|_{t=3} = \left.\frac{dy/dt}{dx/dt}\right\|_{t=3} = 1$. $y(1) = y(3) = 3 + \frac{1}{e} + \int_2^3 \frac{dy}{dt}\, dt = 4$	$3 : \begin{cases} 1 : t = 1 \text{ and } t = 3 \\ 1 : \text{slopes} \\ 1 : y\text{-coordinate} \end{cases}$

Question 4



Let R be the region in the first quadrant bounded by the graph of $y = 2\sqrt{x}$, the horizontal line $y = 6$, and the y-axis, as shown in the figure above.

(a) Find the area of R.

(b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 7$.

(c) Region R is the base of a solid. For each y, where $0 \le y \le 6$, the cross section of the solid taken perpendicular to the y-axis is a rectangle whose height is 3 times the length of its base in region R. Write, but do not evaluate, an integral expression that gives the volume of the solid.

(a) Area $= \int_0^9 (6 - 2\sqrt{x})\,dx = \left(6x - \frac{4}{3}x^{3/2}\right)\Big\vert_{x=0}^{x=9} = 18$	$3: \begin{cases} 1: \text{integrand} \\ 1: \text{antiderivative} \\ 1: \text{answer} \end{cases}$
(b) Volume $= \pi\int_0^9 \left((7 - 2\sqrt{x})^2 - (7 - 6)^2\right) dx$	$3: \begin{cases} 2: \text{integrand} \\ 1: \text{limits and constant} \end{cases}$
(c) Solving $y = 2\sqrt{x}$ for x yields $x = \frac{y^2}{4}$. Each rectangular cross section has area $\left(3\frac{y^2}{4}\right)\left(\frac{y^2}{4}\right) = \frac{3}{16}y^4$. Volume $= \int_0^6 \frac{3}{16}y^4\,dy$	$3: \begin{cases} 2: \text{integrand} \\ 1: \text{answer} \end{cases}$

Question 5

Consider the differential equation $\frac{dy}{dx} = 1 - y$. Let $y = f(x)$ be the particular solution to this differential equation with the initial condition $f(1) = 0$. For this particular solution, $f(x) < 1$ for all values of x.

(a) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(0)$. Show the work that leads to your answer.

(b) Find $\lim_{x \to 1} \frac{f(x)}{x^3 - 1}$. Show the work that leads to your answer.

(c) Find the particular solution $y = f(x)$ to the differential equation $\frac{dy}{dx} = 1 - y$ with the initial condition $f(1) = 0$.

(a) $f\left(\frac{1}{2}\right) \approx f(1) + \left(\left.\frac{dy}{dx}\right\|_{(1,\,0)}\right) \cdot \Delta x$ $= 0 + 1 \cdot \left(-\frac{1}{2}\right) = -\frac{1}{2}$ $f(0) \approx f\left(\frac{1}{2}\right) + \left(\left.\frac{dy}{dx}\right\|_{\left(\frac{1}{2},\,-\frac{1}{2}\right)}\right) \cdot \Delta x$ $\approx -\frac{1}{2} + \frac{3}{2} \cdot \left(-\frac{1}{2}\right) = -\frac{5}{4}$	2 : { 1 : Euler's method with two steps 1 : answer
(b) Since f is differentiable at $x = 1$, f is continuous at $x = 1$. So, $\lim_{x \to 1} f(x) = 0 = \lim_{x \to 1} (x^3 - 1)$ and we may apply L'Hospital's Rule. $\lim_{x \to 1} \frac{f(x)}{x^3 - 1} = \lim_{x \to 1} \frac{f'(x)}{3x^2} = \frac{\lim_{x \to 1} f'(x)}{\lim_{x \to 1} 3x^2} = \frac{1}{3}$	2 : { 1 : use of L'Hospital's Rule 1 : answer
(c) $\frac{dy}{dx} = 1 - y$ $\int \frac{1}{1 - y}\, dy = \int 1\, dx$ $-\ln\|1 - y\| = x + C$ $-\ln 1 = 1 + C \Rightarrow C = -1$ $\ln\|1 - y\| = 1 - x$ $\|1 - y\| = e^{1-x}$ $f(x) = 1 - e^{1-x}$	5 : { 1 : separation of variables 1 : antiderivatives 1 : constant of integration 1 : uses initial condition 1 : solves for y Note: max 2/5 [1-1-0-0-0] if no constant of integration Note: 0/5 if no separation of variables

Question 6

$$f(x) = \begin{cases} \dfrac{\cos x - 1}{x^2} & \text{for } x \neq 0 \\ -\dfrac{1}{2} & \text{for } x = 0 \end{cases}$$

The function f, defined above, has derivatives of all orders. Let g be the function defined by $g(x) = 1 + \int_0^x f(t)\,dt$.

(a) Write the first three nonzero terms and the general term of the Taylor series for $\cos x$ about $x = 0$. Use this series to write the first three nonzero terms and the general term of the Taylor series for f about $x = 0$.

(b) Use the Taylor series for f about $x = 0$ found in part (a) to determine whether f has a relative maximum, relative minimum, or neither at $x = 0$. Give a reason for your answer.

(c) Write the fifth-degree Taylor polynomial for g about $x = 0$.

(d) The Taylor series for g about $x = 0$, evaluated at $x = 1$, is an alternating series with individual terms that decrease in absolute value to 0. Use the third-degree Taylor polynomial for g about $x = 0$ to estimate the value of $g(1)$. Explain why this estimate differs from the actual value of $g(1)$ by less than $\frac{1}{6!}$.

Solution	Scoring
(a) $\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$ $f(x) = -\frac{1}{2} + \frac{x^2}{4!} - \frac{x^4}{6!} + \cdots + (-1)^{n+1} \frac{x^{2n}}{(2n+2)!} + \cdots$	3 : { 1 : terms for $\cos x$ 2 : terms for f 1 : first three terms 1 : general term
(b) $f'(0)$ is the coefficient of x in the Taylor series for f about $x = 0$, so $f'(0) = 0$. $\frac{f''(0)}{2!} = \frac{1}{4!}$ is the coefficient of x^2 in the Taylor series for f about $x = 0$, so $f''(0) = \frac{1}{12}$. Therefore, by the Second Derivative Test, f has a relative minimum at $x = 0$.	2 : { 1 : determines $f'(0)$ 1 : answer with reason
(c) $P_5(x) = 1 - \frac{x}{2} + \frac{x^3}{3 \cdot 4!} - \frac{x^5}{5 \cdot 6!}$	2 : { 1 : two correct terms 1 : remaining terms
(d) $g(1) \approx 1 - \frac{1}{2} + \frac{1}{3 \cdot 4!} = \frac{37}{72}$ Since the Taylor series for g about $x = 0$ evaluated at $x = 1$ is alternating and the terms decrease in absolute value to 0, we know $\left| g(1) - \frac{37}{72} \right| < \frac{1}{5 \cdot 6!} < \frac{1}{6!}$.	2 : { 1 : estimate 1 : explanation

CALCULUS BC
SECTION II, Part A
Time—30 minutes
Number of problems—2

A graphing calculator is required for these problems.

1. A cylindrical can of radius 10 millimeters is used to measure rainfall in Stormville. The can is initially empty, and rain enters the can during a 60-day period. The height of water in the can is modeled by the function S, where $S(t)$ is measured in millimeters and t is measured in days for $0 \le t \le 60$. The rate at which the height of the water is rising in the can is given by $S'(t) = 2\sin(0.03t) + 1.5$.

   (a) According to the model, what is the height of the water in the can at the end of the 60-day period?

   (b) According to the model, what is the average rate of change in the height of water in the can over the 60-day period? Show the computations that lead to your answer. Indicate units of measure.

   (c) Assuming no evaporation occurs, at what rate is the volume of water in the can changing at time $t = 7$? Indicate units of measure.

   (d) During the same 60-day period, rain on Monsoon Mountain accumulates in a can identical to the one in Stormville. The height of the water in the can on Monsoon Mountain is modeled by the function M, where $M(t) = \frac{1}{400}\left(3t^3 - 30t^2 + 330t\right)$. The height $M(t)$ is measured in millimeters, and t is measured in days for $0 \le t \le 60$. Let $D(t) = M'(t) - S'(t)$. Apply the Intermediate Value Theorem to the function D on the interval $0 \le t \le 60$ to justify that there exists a time t, $0 < t < 60$, at which the heights of water in the two cans are changing at the same rate.

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2. The polar curve r is given by $r(\theta) = 3\theta + \sin\theta$, where $0 \le \theta \le 2\pi$.

   (a) Find the area in the second quadrant enclosed by the coordinate axes and the graph of r.

   (b) For $\frac{\pi}{2} \le \theta \le \pi$, there is one point P on the polar curve r with x-coordinate -3. Find the angle θ that corresponds to point P. Find the y-coordinate of point P. Show the work that leads to your answers.

   (c) A particle is traveling along the polar curve r so that its position at time t is $(x(t), y(t))$ and such that $\frac{d\theta}{dt} = 2$. Find $\frac{dy}{dt}$ at the instant that $\theta = \frac{2\pi}{3}$, and interpret the meaning of your answer in the context of the problem.

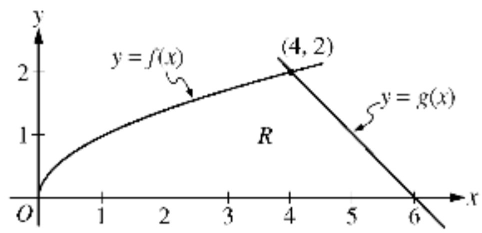
CALCULUS BC

SECTION II, Part B

Time—60 minutes

Number of problems—4

No calculator is allowed for these problems.

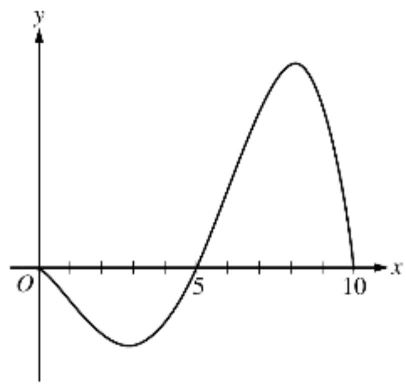


3. The functions f and g are given by $f(x) = \sqrt{x}$ and $g(x) = 6 - x$. Let R be the region bounded by the x-axis and the graphs of f and g, as shown in the figure above.

   (a) Find the area of R.

   (b) The region R is the base of a solid. For each y, where $0 \leq y \leq 2$, the cross section of the solid taken perpendicular to the y-axis is a rectangle whose base lies in R and whose height is $2y$. Write, but do not evaluate, an integral expression that gives the volume of the solid.

   (c) There is a point P on the graph of f at which the line tangent to the graph of f is perpendicular to the graph of g. Find the coordinates of point P.



Graph of f

4. The graph of the differentiable function $y = f(x)$ with domain $0 \le x \le 10$ is shown in the figure above. The area of the region enclosed between the graph of f and the x-axis for $0 \le x \le 5$ is 10, and the area of the region enclosed between the graph of f and the x-axis for $5 \le x \le 10$ is 27. The arc length for the portion of the graph of f between $x = 0$ and $x = 5$ is 11, and the arc length for the portion of the graph of f between $x = 5$ and $x = 10$ is 18. The function f has exactly two critical points that are located at $x = 3$ and $x = 8$.

(a) Find the average value of f on the interval $0 \le x \le 5$.

(b) Evaluate $\int_0^{10} (3f(x) + 2)\, dx$. Show the computations that lead to your answer.

(c) Let $g(x) = \int_5^x f(t)\, dt$. On what intervals, if any, is the graph of g both concave up and decreasing? Explain your reasoning.

(d) The function h is defined by $h(x) = 2f\left(\frac{x}{2}\right)$. The derivative of h is $h'(x) = f'\left(\frac{x}{2}\right)$. Find the arc length of the graph of $y = h(x)$ from $x = 0$ to $x = 20$.

t (seconds)	0	10	40	60
$B(t)$ (meters)	100	136	9	49
$v(t)$ (meters per second)	2.0	2.3	2.5	4.6

5. Ben rides a unicycle back and forth along a straight east-west track. The twice-differentiable function B models Ben's position on the track, measured in meters from the western end of the track, at time t, measured in seconds from the start of the ride. The table above gives values for $B(t)$ and Ben's velocity, $v(t)$, measured in meters per second, at selected times t.

(a) Use the data in the table to approximate Ben's acceleration at time $t = 5$ seconds. Indicate units of measure.

(b) Using correct units, interpret the meaning of $\int_0^{60} |v(t)|\,dt$ in the context of this problem. Approximate $\int_0^{60} |v(t)|\,dt$ using a left Riemann sum with the subintervals indicated by the data in the table.

(c) For $40 \le t \le 60$, must there be a time t when Ben's velocity is 2 meters per second? Justify your answer.

(d) A light is directly above the western end of the track. Ben rides so that at time t, the distance $L(t)$ between Ben and the light satisfies $(L(t))^2 = 12^2 + (B(t))^2$. At what rate is the distance between Ben and the light changing at time $t = 40$?

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6. Let $f(x) = \ln\left(1 + x^3\right)$.

(a) The Maclaurin series for $\ln(1 + x)$ is $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1} \cdot \frac{x^n}{n} + \cdots$. Use the series to write the first four nonzero terms and the general term of the Maclaurin series for f.

(b) The radius of convergence of the Maclaurin series for f is 1. Determine the interval of convergence. Show the work that leads to your answer.

(c) Write the first four nonzero terms of the Maclaurin series for $f'\left(t^2\right)$. If $g(x) = \int_0^x f'\left(t^2\right)dt$, use the first two nonzero terms of the Maclaurin series for g to approximate $g(1)$.

(d) The Maclaurin series for g, evaluated at $x = 1$, is a convergent alternating series with individual terms that decrease in absolute value to 0. Show that your approximation in part (c) must differ from $g(1)$ by less than $\frac{1}{5}$.

Question 1

A cylindrical can of radius 10 millimeters is used to measure rainfall in Stormville. The can is initially empty, and rain enters the can during a 60-day period. The height of water in the can is modeled by the function S, where $S(t)$ is measured in millimeters and t is measured in days for $0 \le t \le 60$. The rate at which the height of the water is rising in the can is given by $S'(t) = 2\sin(0.03t) + 1.5$.

(a) According to the model, what is the height of the water in the can at the end of the 60-day period?

(b) According to the model, what is the average rate of change in the height of water in the can over the 60-day period? Show the computations that lead to your answer. Indicate units of measure.

(c) Assuming no evaporation occurs, at what rate is the volume of water in the can changing at time $t = 7$? Indicate units of measure.

(d) During the same 60-day period, rain on Monsoon Mountain accumulates in a can identical to the one in Stormville. The height of the water in the can on Monsoon Mountain is modeled by the function M, where $M(t) = \frac{1}{400}\left(3t^3 - 30t^2 + 330t\right)$. The height $M(t)$ is measured in millimeters, and t is measured in days for $0 \le t \le 60$. Let $D(t) = M'(t) - S'(t)$. Apply the Intermediate Value Theorem to the function D on the interval $0 \le t \le 60$ to justify that there exists a time t, $0 < t < 60$, at which the heights of water in the two cans are changing at the same rate.

Solution	Scoring
(a) $S(60) = \int_0^{60} S'(t)\,dt = 171.813$ mm	$3: \begin{cases} 1: \text{limits} \\ 1: \text{integrand} \\ 1: \text{answer} \end{cases}$
(b) $\dfrac{S(60) - S(0)}{60} = 2.863$ or 2.864 mm/day	1 : answer
(c) $V(t) = 100\pi S(t)$ $V'(7) = 100\pi S'(7) = 602.218$ The volume of water in the can is increasing at a rate of $602.218\ \text{mm}^3/\text{day}$.	$2: \begin{cases} 1: \text{relationship between } V \text{ and } S \\ 1: \text{answer} \end{cases}$
(d) $D(0) = -0.675 < 0$ and $D(60) = 69.37730 > 0$ Because D is continuous, the Intermediate Value Theorem implies that there is a time t, $0 < t < 60$, at which $D(t) = 0$. At this time, the heights of water in the two cans are changing at the same rate.	$2: \begin{cases} 1: \text{considers } D(0) \text{ and } D(60) \\ 1: \text{justification} \end{cases}$
	1 : units in (b) or (c)

Question 2

The polar curve r is given by $r(\theta) = 3\theta + \sin\theta$, where $0 \le \theta \le 2\pi$.

(a) Find the area in the second quadrant enclosed by the coordinate axes and the graph of r.

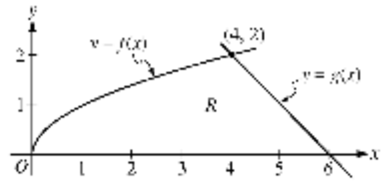
(b) For $\frac{\pi}{2} \le \theta \le \pi$, there is one point P on the polar curve r with x-coordinate -3. Find the angle θ that corresponds to point P. Find the y-coordinate of point P. Show the work that leads to your answers.

(c) A particle is traveling along the polar curve r so that its position at time t is $(x(t), y(t))$ and such that $\frac{d\theta}{dt} = 2$. Find $\frac{dy}{dt}$ at the instant that $\theta = \frac{2\pi}{3}$, and interpret the meaning of your answer in the context of the problem.

(a) Area $= \frac{1}{2}\int_{\pi/2}^{\pi} (r(\theta))^2 \, d\theta = 47.513$	$3: \begin{cases} 1: \text{integrand} \\ 1: \text{limits and constant} \\ 1: \text{answer} \end{cases}$
(b) $-3 = r(\theta)\cos\theta = (3\theta + \sin\theta)\cos\theta$ $\theta = 2.01692$ $y = r(\theta)\sin(\theta) = 6.272$	$3: \begin{cases} 1: \text{equation} \\ 1: \text{value of } \theta \\ 1: y\text{-coordinate} \end{cases}$
(c) $y = r(\theta)\sin\theta = (3\theta + \sin\theta)\sin\theta$ $\left.\frac{dy}{dt}\right\|_{\theta=2\pi/3} = \left[\frac{dy}{d\theta}\cdot\frac{d\theta}{dt}\right]_{\theta=2\pi/3} = -2.819$ The y-coordinate of the particle is decreasing at a rate of 2.819.	$3: \begin{cases} 1: \text{uses chain rule} \\ 1: \text{answer} \\ 1: \text{interpretation} \end{cases}$

Question 3

The functions f and g are given by $f(x) = \sqrt{x}$ and $g(x) = 6 - x$. Let R be the region bounded by the x-axis and the graphs of f and g, as shown in the figure above.



(a) Find the area of R.

(b) The region R is the base of a solid. For each y, where $0 \le y \le 2$, the cross section of the solid taken perpendicular to the y-axis is a rectangle whose base lies in R and whose height is $2y$. Write, but do not evaluate, an integral expression that gives the volume of the solid.

(c) There is a point P on the graph of f at which the line tangent to the graph of f is perpendicular to the graph of g. Find the coordinates of point P.

(a) Area $= \int_0^4 \sqrt{x}\,dx + \frac{1}{2}\cdot 2\cdot 2 = \frac{2}{3}x^{3/2}\Big|_{x=0}^{x=4} + 2 = \frac{22}{3}$

$3: \begin{cases} 1: \text{integral} \\ 1: \text{antiderivative} \\ 1: \text{answer} \end{cases}$

(b) $y = \sqrt{x} \Rightarrow x = y^2$

$y = 6 - x \Rightarrow x = 6 - y$

Width $= (6 - y) - y^2$

Volume $= \int_0^2 2y\left(6 - y - y^2\right)dy$

$3: \begin{cases} 2: \text{integrand} \\ 1: \text{answer} \end{cases}$

(c) $g'(x) = -1$

Thus a line perpendicular to the graph of g has slope 1.

$f'(x) = \frac{1}{2\sqrt{x}}$

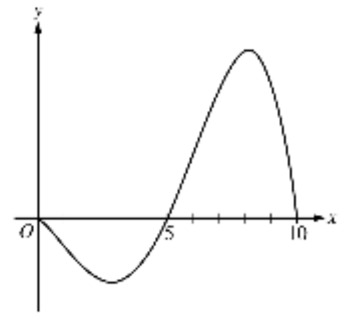
$\frac{1}{2\sqrt{x}} = 1 \Rightarrow x = \frac{1}{4}$

The point P has coordinates $\left(\frac{1}{4}, \frac{1}{2}\right)$.

$3: \begin{cases} 1: f'(x) \\ 1: \text{equation} \\ 1: \text{answer} \end{cases}$

Question 4

The graph of the differentiable function $y = f(x)$ with domain $0 \le x \le 10$ is shown in the figure above. The area of the region enclosed between the graph of f and the x-axis for $0 \le x \le 5$ is 10, and the area of the region enclosed between the graph of f and the x-axis for $5 \le x \le 10$ is 27. The arc length for the portion of the graph of f between $x = 0$ and $x = 5$ is 11, and the arc length for the portion of the graph of f between $x = 5$ and $x = 10$ is 18. The function f has exactly two critical points that are located at $x = 3$ and $x = 8$.



Graph of f

(a) Find the average value of f on the interval $0 \le x \le 5$.

(b) Evaluate $\int_0^{10} (3f(x) + 2)\,dx$. Show the computations that lead to your answer.

(c) Let $g(x) = \int_5^x f(t)\,dt$. On what intervals, if any, is the graph of g both concave up and decreasing? Explain your reasoning.

(d) The function h is defined by $h(x) = 2f\left(\frac{x}{2}\right)$. The derivative of h is $h'(x) = f'\left(\frac{x}{2}\right)$. Find the arc length of the graph of $y = h(x)$ from $x = 0$ to $x = 20$.

Solution	Scoring
(a) Average value $= \frac{1}{5}\int_0^5 f(x)\,dx = \frac{-10}{5} = -2$	1 : answer
(b) $\int_0^{10} (3f(x) + 2)\,dx = 3\left(\int_0^5 f(x)\,dx + \int_5^{10} f(x)\,dx\right) + 20$ $= 3(-10 + 27) + 20 = 71$	2 : answer
(c) $g'(x) = f(x)$ $g'(x) < 0$ on $0 < x < 5$ $g'(x)$ is increasing on $3 < x < 8$. The graph of g is concave up and decreasing on $3 < x < 5$.	3 : { 1 : $g'(x) = f(x)$ 1 : analysis 1 : answer and reason
(d) Arc length $= \int_0^{20} \sqrt{1 + (h'(x))^2}\,dx = \int_0^{20} \sqrt{1 + \left(f'\left(\frac{x}{2}\right)\right)^2}\,dx$ Let $u = \frac{x}{2}$. Then $du = \frac{1}{2}dx$ and $\int_0^{20} \sqrt{1 + \left(f'\left(\frac{x}{2}\right)\right)^2}\,dx = 2\int_0^{10} \sqrt{1 + (f'(u))^2}\,du = 2(11 + 18) = 58$	3 : { 1 : integral 1 : substitution 1 : answer

Question 5

t (seconds)	0	10	40	60
$B(t)$ (meters)	100	136	9	49
$v(t)$ (meters per second)	2.0	2.3	2.5	4.6

Ben rides a unicycle back and forth along a straight east-west track. The twice-differentiable function B models Ben's position on the track, measured in meters from the western end of the track, at time t, measured in seconds from the start of the ride. The table above gives values for $B(t)$ and Ben's velocity, $v(t)$, measured in meters per second, at selected times t.

(a) Use the data in the table to approximate Ben's acceleration at time $t = 5$ seconds. Indicate units of measure.

(b) Using correct units, interpret the meaning of $\int_0^{60} |v(t)|\,dt$ in the context of this problem. Approximate $\int_0^{60} |v(t)|\,dt$ using a left Riemann sum with the subintervals indicated by the data in the table.

(c) For $40 \le t \le 60$, must there be a time t when Ben's velocity is 2 meters per second? Justify your answer.

(d) A light is directly above the western end of the track. Ben rides so that at time t, the distance $L(t)$ between Ben and the light satisfies $(L(t))^2 = 12^2 + (B(t))^2$. At what rate is the distance between Ben and the light changing at time $t = 40$?

Solution	Scoring
(a) $a(5) \approx \dfrac{v(10) - v(0)}{10 - 0} = \dfrac{0.3}{10} = 0.03$ meters/sec^2	1 : answer
(b) $\int_0^{60} \lvert v(t)\rvert\,dt$ is the total distance, in meters, Ben rides over the 60-second interval $t = 0$ to $t = 60$. $\int_0^{60} \lvert v(t)\rvert\,dt \approx 2.0 \cdot 10 + 2.3(40 - 10) + 2.5(60 - 40) = 139$ meters	2 : { 1 : meaning of integral; 1 : approximation }
(c) Because $\dfrac{B(60) - B(40)}{60 - 40} = \dfrac{49 - 9}{20} = 2$, the Mean Value Theorem implies there is a time t, $40 < t < 60$, such that $v(t) = 2$.	2 : { 1 : difference quotient; 1 : conclusion with justification }
(d) $2L(t)L'(t) = 2B(t)B'(t)$ $L'(40) = \dfrac{B(40)v(40)}{L(40)} = \dfrac{9 \cdot 2.5}{\sqrt{144 + 81}} = \dfrac{3}{2}$ meters/sec	3 : { 1 : derivatives; 1 : uses $B'(t) = v(t)$; 1 : answer }
	1 : units in (a) or (b)

Question 6

Let $f(x) = \ln\left(1 + x^3\right)$.

(a) The Maclaurin series for $\ln(1 + x)$ is $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1} \cdot \frac{x^n}{n} + \cdots$. Use the series to write the first four nonzero terms and the general term of the Maclaurin series for f.

(b) The radius of convergence of the Maclaurin series for f is 1. Determine the interval of convergence. Show the work that leads to your answer.

(c) Write the first four nonzero terms of the Maclaurin series for $f'\left(t^2\right)$. If $g(x) = \int_0^x f'\left(t^2\right)\,dt$, use the first two nonzero terms of the Maclaurin series for g to approximate $g(1)$.

(d) The Maclaurin series for g, evaluated at $x = 1$, is a convergent alternating series with individual terms that decrease in absolute value to 0. Show that your approximation in part (c) must differ from $g(1)$ by less than $\frac{1}{5}$.

(a) $x^3 - \frac{x^6}{2} + \frac{x^9}{3} - \frac{x^{12}}{4} + \cdots + (-1)^{n+1} \cdot \frac{x^{3n}}{n} + \cdots$

$2 : \begin{cases} 1 : \text{first four terms} \\ 1 : \text{general term} \end{cases}$

(b) The interval of convergence is centered at $x = 0$.
At $x = -1$, the series is $-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots - \frac{1}{n} - \cdots$, which diverges because the harmonic series diverges.
At $x = 1$, the series is $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n+1} \cdot \frac{1}{n} + \cdots$, the alternating harmonic series, which converges.

Therefore the interval of convergence is $-1 < x \le 1$.

2 : answer with analysis

(c) The Maclaurin series for $f'(x)$, $f'\left(t^2\right)$, and $g(x)$ are

$f'(x) : \sum_{n=1}^{\infty} (-1)^{n+1} \cdot 3x^{3n-1} = 3x^2 - 3x^5 + 3x^8 - 3x^{11} + \cdots$

$f'\left(t^2\right) : \sum_{n=1}^{\infty} (-1)^{n+1} \cdot 3t^{6n-2} = 3t^4 - 3t^{10} + 3t^{16} - 3t^{22} + \cdots$

$g(x) : \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{3x^{6n-1}}{6n-1} = \frac{3x^5}{5} - \frac{3x^{11}}{11} + \frac{3x^{17}}{17} - \frac{3x^{23}}{23} + \cdots$

Thus $g(1) \approx \frac{3}{5} - \frac{3}{11} = \frac{18}{55}$.

$4 : \begin{cases} 1 : \text{two terms for } f'\left(t^2\right) \\ 1 : \text{other terms for } f'\left(t^2\right) \\ 1 : \text{first two terms for } g(x) \\ 1 : \text{approximation} \end{cases}$

(d) The Maclaurin series for g evaluated at $x = 1$ is alternating, and the terms decrease in absolute value to 0.
Thus $\left|g(1) - \frac{18}{55}\right| < \frac{3 \cdot 1^{17}}{17} = \frac{3}{17} < \frac{1}{5}$.

1 : analysis

CALCULUS BC
SECTION II, Part A
Time—30 minutes
Number of problems—2

A graphing calculator is required for these problems.

1. At time t, a particle moving in the xy-plane is at position $(x(t), y(t))$, where $x(t)$ and $y(t)$ are not explicitly given. For $t \geq 0$, $\frac{dx}{dt} = 4t + 1$ and $\frac{dy}{dt} = \sin\left(t^2\right)$. At time $t = 0$, $x(0) = 0$ and $y(0) = -4$.

   (a) Find the speed of the particle at time $t = 3$, and find the acceleration vector of the particle at time $t = 3$.

   (b) Find the slope of the line tangent to the path of the particle at time $t = 3$.

   (c) Find the position of the particle at time $t = 3$.

   (d) Find the total distance traveled by the particle over the time interval $0 \leq t \leq 3$.

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t (minutes)	0	2	5	9	10
$H(t)$ (degrees Celsius)	66	60	52	44	43

2. As a pot of tea cools, the temperature of the tea is modeled by a differentiable function H for $0 \leq t \leq 10$, where time t is measured in minutes and temperature $H(t)$ is measured in degrees Celsius. Values of $H(t)$ at selected values of time t are shown in the table above.

   (a) Use the data in the table to approximate the rate at which the temperature of the tea is changing at time $t = 3.5$. Show the computations that lead to your answer.

   (b) Using correct units, explain the meaning of $\frac{1}{10}\int_0^{10} H(t)\,dt$ in the context of this problem. Use a trapezoidal sum with the four subintervals indicated by the table to estimate $\frac{1}{10}\int_0^{10} H(t)\,dt$.

   (c) Evaluate $\int_0^{10} H'(t)\,dt$. Using correct units, explain the meaning of the expression in the context of this problem.

   (d) At time $t = 0$, biscuits with temperature 100°C were removed from an oven. The temperature of the biscuits at time t is modeled by a differentiable function B for which it is known that $B'(t) = -13.84e^{-0.173t}$. Using the given models, at time $t = 10$, how much cooler are the biscuits than the tea?

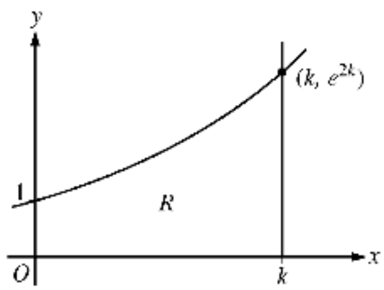
CALCULUS BC

SECTION II, Part B

Time—60 minutes

Number of problems—4

No calculator is allowed for these problems.

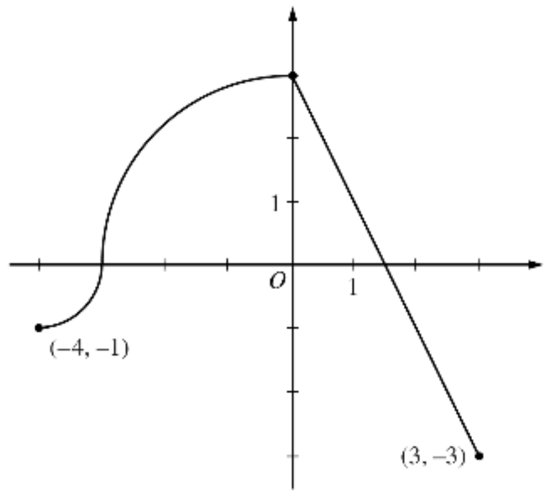


3. Let $f(x) = e^{2x}$. Let R be the region in the first quadrant bounded by the graph of f, the coordinate axes, and the vertical line $x = k$, where $k > 0$. The region R is shown in the figure above.

   (a) Write, but do not evaluate, an expression involving an integral that gives the perimeter of R in terms of k.

   (b) The region R is rotated about the x-axis to form a solid. Find the volume, V, of the solid in terms of k.

   (c) The volume V, found in part (b), changes as k changes. If $\frac{dk}{dt} = \frac{1}{3}$, determine $\frac{dV}{dt}$ when $k = \frac{1}{2}$.



Graph of f

4. The continuous function f is defined on the interval $-4 \le x \le 3$. The graph of f consists of two quarter circles and one line segment, as shown in the figure above. Let $g(x) = 2x + \int_0^x f(t)\,dt$.

   (a) Find $g(-3)$. Find $g'(x)$ and evaluate $g'(-3)$.

   (b) Determine the x-coordinate of the point at which g has an absolute maximum on the interval $-4 \le x \le 3$. Justify your answer.

   (c) Find all values of x on the interval $-4 < x < 3$ for which the graph of g has a point of inflection. Give a reason for your answer.

   (d) Find the average rate of change of f on the interval $-4 \le x \le 3$. There is no point c, $-4 < c < 3$, for which $f'(c)$ is equal to that average rate of change. Explain why this statement does not contradict the Mean Value Theorem.

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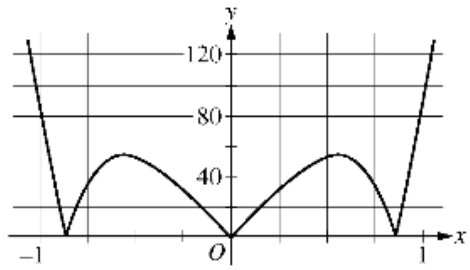
5. At the beginning of 2010, a landfill contained 1400 tons of solid waste. The increasing function W models the total amount of solid waste stored at the landfill. Planners estimate that W will satisfy the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ for the next 20 years. W is measured in tons, and t is measured in years from the start of 2010.

   (a) Use the line tangent to the graph of W at $t = 0$ to approximate the amount of solid waste that the landfill contains at the end of the first 3 months of 2010 (time $t = \frac{1}{4}$).

   (b) Find $\frac{d^2W}{dt^2}$ in terms of W. Use $\frac{d^2W}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the amount of solid waste that the landfill contains at time $t = \frac{1}{4}$.

   (c) Find the particular solution $W = W(t)$ to the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ with initial condition $W(0) = 1400$.



Graph of $y = \left|f^{(5)}(x)\right|$

6. Let $f(x) = \sin\left(x^2\right) + \cos x$. The graph of $y = \left|f^{(5)}(x)\right|$ is shown above.

(a) Write the first four nonzero terms of the Taylor series for $\sin x$ about $x = 0$, and write the first four nonzero terms of the Taylor series for $\sin\left(x^2\right)$ about $x = 0$.

(b) Write the first four nonzero terms of the Taylor series for $\cos x$ about $x = 0$. Use this series and the series for $\sin\left(x^2\right)$, found in part (a), to write the first four nonzero terms of the Taylor series for f about $x = 0$.

(c) Find the value of $f^{(6)}(0)$.

(d) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Using information from the graph of $y = \left|f^{(5)}(x)\right|$ shown above, show that $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| < \frac{1}{3000}$.

Question 1

At time t, a particle moving in the xy-plane is at position $(x(t), y(t))$, where $x(t)$ and $y(t)$ are not explicitly given. For $t \geq 0$, $\frac{dx}{dt} = 4t + 1$ and $\frac{dy}{dt} = \sin(t^2)$. At time $t = 0$, $x(0) = 0$ and $y(0) = -4$.

(a) Find the speed of the particle at time $t = 3$, and find the acceleration vector of the particle at time $t = 3$.

(b) Find the slope of the line tangent to the path of the particle at time $t = 3$.

(c) Find the position of the particle at time $t = 3$.

(d) Find the total distance traveled by the particle over the time interval $0 \leq t \leq 3$.

Solution	Scoring
(a) Speed $= \sqrt{(x'(3))^2 + (y'(3))^2} = 13.006$ or 13.007 Acceleration $= \langle x''(3), y''(3)\rangle = \langle 4, -5.466\rangle$ or $\langle 4, -5.467\rangle$	2 : { 1 : speed; 1 : acceleration
(b) Slope $= \frac{y'(3)}{x'(3)} = 0.031$ or 0.032	1 : answer
(c) $x(3) = 0 + \int_0^3 \frac{dx}{dt}\,dt = 21$ $y(3) = -4 + \int_0^3 \frac{dy}{dt}\,dt = -3.226$ At time $t = 3$, the particle is at position $(21, -3.226)$.	4 : { 2 : x-coordinate (1 : integral; 1 : answer); 2 : y-coordinate (1 : integral; 1 : answer)
(d) Distance $= \int_0^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = 21.091$	2 : { 1 : integral; 1 : answer

Question 2

t (minutes)	0	2	5	9	10
$H(t)$ (degrees Celsius)	66	60	52	44	43

As a pot of tea cools, the temperature of the tea is modeled by a differentiable function H for $0 \le t \le 10$, where time t is measured in minutes and temperature $H(t)$ is measured in degrees Celsius. Values of $H(t)$ at selected values of time t are shown in the table above.

(a) Use the data in the table to approximate the rate at which the temperature of the tea is changing at time $t = 3.5$. Show the computations that lead to your answer.

(b) Using correct units, explain the meaning of $\frac{1}{10}\int_0^{10} H(t)\,dt$ in the context of this problem. Use a trapezoidal sum with the four subintervals indicated by the table to estimate $\frac{1}{10}\int_0^{10} H(t)\,dt$.

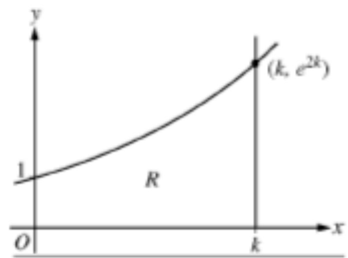
(c) Evaluate $\int_0^{10} H'(t)\,dt$. Using correct units, explain the meaning of the expression in the context of this problem.

(d) At time $t = 0$, biscuits with temperature 100°C were removed from an oven. The temperature of the biscuits at time t is modeled by a differentiable function B for which it is known that $B'(t) = -13.84e^{-0.173t}$. Using the given models, at time $t = 10$, how much cooler are the biscuits than the tea?

Solution	Scoring
(a) $H'(3.5) \approx \dfrac{H(5) - H(2)}{5 - 2} = \dfrac{52 - 60}{3} = -2.666$ or -2.667 degrees Celsius per minute	1 : answer
(b) $\frac{1}{10}\int_0^{10} H(t)\,dt$ is the average temperature of the tea, in degrees Celsius, over the 10 minutes. $\frac{1}{10}\int_0^{10} H(t)\,dt \approx \frac{1}{10}\left(2 \cdot \frac{66+60}{2} + 3 \cdot \frac{60+52}{2} + 4 \cdot \frac{52+44}{2} + 1 \cdot \frac{44+43}{2}\right) = 52.95$	3 : { 1 : meaning of expression; 1 : trapezoidal sum; 1 : estimate }
(c) $\int_0^{10} H'(t)\,dt = H(10) - H(0) = 43 - 66 = -23$. The temperature of the tea drops 23 degrees Celsius from time $t = 0$ to time $t = 10$ minutes.	2 : { 1 : value of integral; 1 : meaning of expression }
(d) $B(10) = 100 + \int_0^{10} B'(t)\,dt = 34.18275$; $H(10) - B(10) = 8.817$. The biscuits are 8.817 degrees Celsius cooler than the tea.	3 : { 1 : integrand; 1 : uses $B(0) = 100$; 1 : answer }

Question 3

Let $f(x) = e^{2x}$. Let R be the region in the first quadrant bounded by the graph of f, the coordinate axes, and the vertical line $x = k$, where $k > 0$. The region R is shown in the figure above.



(a) Write, but do not evaluate, an expression involving an integral that gives the perimeter of R in terms of k.

(b) The region R is rotated about the x-axis to form a solid. Find the volume, V, of the solid in terms of k.

(c) The volume V, found in part (b), changes as k changes. If $\frac{dk}{dt} = \frac{1}{3}$, determine $\frac{dV}{dt}$ when $k = \frac{1}{2}$.

(a) $f'(x) = 2e^{2x}$

Perimeter $= 1 + k + e^{2k} + \int_0^k \sqrt{1 + \left(2e^{2x}\right)^2}\, dx$

$3 : \begin{cases} 1 : f'(x) \\ 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) Volume $= \pi \int_0^k \left(e^{2x}\right)^2 dx = \pi \int_0^k e^{4x}\, dx = \frac{\pi}{4} e^{4x} \Big|_{x=0}^{x=k} = \frac{\pi}{4} e^{4k} - \frac{\pi}{4}$

$4 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

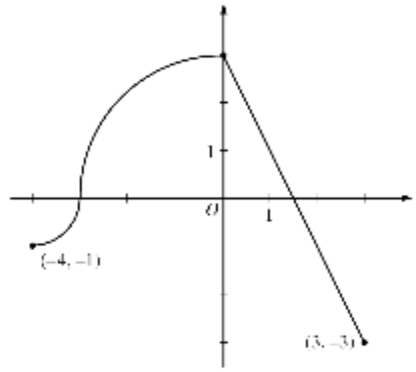
(c) $\frac{dV}{dt} = \pi e^{4k} \frac{dk}{dt}$

When $k = \frac{1}{2}$, $\frac{dV}{dt} = \pi e^2 \cdot \frac{1}{3}$.

$2 : \begin{cases} 1 : \text{applies chain rule} \\ 1 : \text{answer} \end{cases}$

Question 4

The continuous function f is defined on the interval $-4 \le x \le 3$. The graph of f consists of two quarter circles and one line segment, as shown in the figure above.



Graph of f

Let $g(x) = 2x + \int_0^x f(t)\,dt$.

(a) Find $g(-3)$. Find $g'(x)$ and evaluate $g'(-3)$.

(b) Determine the x-coordinate of the point at which g has an absolute maximum on the interval $-4 \le x \le 3$. Justify your answer.

(c) Find all values of x on the interval $-4 < x < 3$ for which the graph of g has a point of inflection. Give a reason for your answer.

(d) Find the average rate of change of f on the interval $-4 \le x \le 3$. There is no point c, $-4 < c < 3$, for which $f'(c)$ is equal to that average rate of change. Explain why this statement does not contradict the Mean Value Theorem.

(a) $g(-3) = 2(-3) + \int_0^{-3} f(t)\,dt = -6 - \frac{9\pi}{4}$

$g'(x) = 2 + f(x)$

$g'(-3) = 2 + f(-3) = 2$

$3 : \begin{cases} 1 : g(-3) \\ 1 : g'(x) \\ 1 : g'(-3) \end{cases}$

(b) $g'(x) = 0$ when $f(x) = -2$. This occurs at $x = \frac{5}{2}$.

$g'(x) > 0$ for $-4 < x < \frac{5}{2}$ and $g'(x) < 0$ for $\frac{5}{2} < x < 3$.

Therefore g has an absolute maximum at $x = \frac{5}{2}$.

$3 : \begin{cases} 1 : \text{considers } g'(x) = 0 \\ 1 : \text{identifies interior candidate} \\ 1 : \text{answer with justification} \end{cases}$

(c) $g''(x) = f'(x)$ changes sign only at $x = 0$. Thus the graph of g has a point of inflection at $x = 0$.

1 : answer with reason

(d) The average rate of change of f on the interval $-4 \le x \le 3$ is $\frac{f(3) - f(-4)}{3 - (-4)} = -\frac{2}{7}$.

To apply the Mean Value Theorem, f must be differentiable at each point in the interval $-4 < x < 3$. However, f is not differentiable at $x = -3$ and $x = 0$.

$2 : \begin{cases} 1 : \text{average rate of change} \\ 1 : \text{explanation} \end{cases}$

Question 5

At the beginning of 2010, a landfill contained 1400 tons of solid waste. The increasing function W models the total amount of solid waste stored at the landfill. Planners estimate that W will satisfy the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ for the next 20 years. W is measured in tons, and t is measured in years from the start of 2010.

(a) Use the line tangent to the graph of W at $t = 0$ to approximate the amount of solid waste that the landfill contains at the end of the first 3 months of 2010 (time $t = \frac{1}{4}$).

(b) Find $\frac{d^2W}{dt^2}$ in terms of W. Use $\frac{d^2W}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the amount of solid waste that the landfill contains at time $t = \frac{1}{4}$.

(c) Find the particular solution $W = W(t)$ to the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ with initial condition $W(0) = 1400$.

(a) $\left.\frac{dW}{dt}\right|_{t=0} = \frac{1}{25}(W(0) - 300) = \frac{1}{25}(1400 - 300) = 44$

The tangent line is $y = 1400 + 44t$.

$W\left(\frac{1}{4}\right) \approx 1400 + 44\left(\frac{1}{4}\right) = 1411$ tons

$2 : \begin{cases} 1 : \frac{dW}{dt} \text{ at } t = 0 \\ 1 : \text{answer} \end{cases}$

(b) $\frac{d^2W}{dt^2} = \frac{1}{25}\frac{dW}{dt} = \frac{1}{625}(W - 300)$ and $W \ge 1400$

Therefore $\frac{d^2W}{dt^2} > 0$ on the interval $0 \le t \le \frac{1}{4}$.

The answer in part (a) is an underestimate.

$2 : \begin{cases} 1 : \frac{d^2W}{dt^2} \\ 1 : \text{answer with reason} \end{cases}$

(c) $\frac{dW}{dt} = \frac{1}{25}(W - 300)$

$\int \frac{1}{W - 300}\, dW = \int \frac{1}{25}\, dt$

$\ln|W - 300| = \frac{1}{25}t + C$

$\ln(1400 - 300) = \frac{1}{25}(0) + C \Rightarrow \ln(1100) = C$

$W - 300 = 1100e^{\frac{1}{25}t}$

$W(t) = 300 + 1100e^{\frac{1}{25}t}, \ \ 0 \le t \le 20$

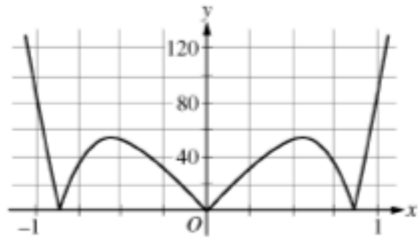
$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } W \end{cases}$

Note: max 2/5 [1-1-0-0-0] if no constant of integration

Note: 0/5 if no separation of variables

Question 6

Let $f(x) = \sin\left(x^2\right) + \cos x$. The graph of $y = \left|f^{(5)}(x)\right|$ is shown above.



Graph of $y = \left|f^{(5)}(x)\right|$

(a) Write the first four nonzero terms of the Taylor series for $\sin x$ about $x = 0$, and write the first four nonzero terms of the Taylor series for $\sin\left(x^2\right)$ about $x = 0$.

(b) Write the first four nonzero terms of the Taylor series for $\cos x$ about $x = 0$. Use this series and the series for $\sin\left(x^2\right)$, found in part (a), to write the first four nonzero terms of the Taylor series for f about $x = 0$.

(c) Find the value of $f^{(6)}(0)$.

(d) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Using information from the graph of $y = \left|f^{(5)}(x)\right|$ shown above, show that $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| < \frac{1}{3000}$.

(a) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$\sin\left(x^2\right) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$

3 : { 1 : series for $\sin x$; 2 : series for $\sin\left(x^2\right)$ }

(b) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$f(x) = 1 + \frac{x^2}{2} + \frac{x^4}{4!} - \frac{121x^6}{6!} + \cdots$

3 : { 1 : series for $\cos x$; 2 : series for $f(x)$ }

(c) $\frac{f^{(6)}(0)}{6!}$ is the coefficient of x^6 in the Taylor series for f about $x = 0$. Therefore $f^{(6)}(0) = -121$.

1 : answer

(d) The graph of $y = \left|f^{(5)}(x)\right|$ indicates that $\max\limits_{0 \le x \le \frac{1}{4}} \left|f^{(5)}(x)\right| < 40$.

Therefore

$$\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| \le \frac{\max\limits_{0 \le x \le \frac{1}{4}} \left|f^{(5)}(x)\right|}{5!} \cdot \left(\frac{1}{4}\right)^5 < \frac{40}{120 \cdot 4^5} = \frac{1}{3072} < \frac{1}{3000}.$$

2 : { 1 : form of the error bound; 1 : analysis }

CALCULUS BC
SECTION II, Part A
Time—30 minutes
Number of problems—2

A graphing calculator is required for these problems.

t (minutes)	0	4	9	15	20
$W(t)$ (degrees Fahrenheit)	55.0	57.1	61.8	67.9	71.0

1. The temperature of water in a tub at time t is modeled by a strictly increasing, twice-differentiable function W, where $W(t)$ is measured in degrees Fahrenheit and t is measured in minutes. At time $t = 0$, the temperature of the water is $55°F$. The water is heated for 30 minutes, beginning at time $t = 0$. Values of $W(t)$ at selected times t for the first 20 minutes are given in the table above.

   (a) Use the data in the table to estimate $W'(12)$. Show the computations that lead to your answer. Using correct units, interpret the meaning of your answer in the context of this problem.

   (b) Use the data in the table to evaluate $\int_0^{20} W'(t)\,dt$. Using correct units, interpret the meaning of $\int_0^{20} W'(t)\,dt$ in the context of this problem.

   (c) For $0 \le t \le 20$, the average temperature of the water in the tub is $\frac{1}{20}\int_0^{20} W(t)\,dt$. Use a left Riemann sum with the four subintervals indicated by the data in the table to approximate $\frac{1}{20}\int_0^{20} W(t)\,dt$. Does this approximation overestimate or underestimate the average temperature of the water over these 20 minutes? Explain your reasoning.

   (d) For $20 \le t \le 25$, the function W that models the water temperature has first derivative given by $W'(t) = 0.4\sqrt{t}\cos(0.06t)$. Based on the model, what is the temperature of the water at time $t = 25$?

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2. For $t \ge 0$, a particle is moving along a curve so that its position at time t is $(x(t), y(t))$. At time $t = 2$, the particle is at position $(1, 5)$. It is known that $\frac{dx}{dt} = \frac{\sqrt{t+2}}{e^t}$ and $\frac{dy}{dt} = \sin^2 t$.

   (a) Is the horizontal movement of the particle to the left or to the right at time $t = 2$? Explain your answer. Find the slope of the path of the particle at time $t = 2$.

   (b) Find the x-coordinate of the particle's position at time $t = 4$.

   (c) Find the speed of the particle at time $t = 4$. Find the acceleration vector of the particle at time $t = 4$.

   (d) Find the distance traveled by the particle from time $t = 2$ to $t = 4$.

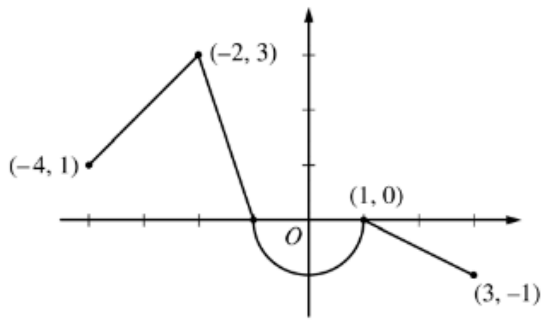
CALCULUS BC

SECTION II, Part B

Time—60 minutes

Number of problems—4

No calculator is allowed for these problems.



Graph of f

3. Let f be the continuous function defined on $[-4, 3]$ whose graph, consisting of three line segments and a semicircle centered at the origin, is given above. Let g be the function given by $g(x) = \int_1^x f(t)\,dt$.

   (a) Find the values of $g(2)$ and $g(-2)$.

   (b) For each of $g'(-3)$ and $g''(-3)$, find the value or state that it does not exist.

   (c) Find the x-coordinate of each point at which the graph of g has a horizontal tangent line. For each of these points, determine whether g has a relative minimum, relative maximum, or neither a minimum nor a maximum at the point. Justify your answers.

   (d) For $-4 < x < 3$, find all values of x for which the graph of g has a point of inflection. Explain your reasoning.

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x	1	1.1	1.2	1.3	1.4
$f'(x)$	8	10	12	13	14.5

4. The function f is twice differentiable for $x > 0$ with $f(1) = 15$ and $f''(1) = 20$. Values of f', the derivative of f, are given for selected values of x in the table above.

   (a) Write an equation for the line tangent to the graph of f at $x = 1$. Use this line to approximate $f(1.4)$.

   (b) Use a midpoint Riemann sum with two subintervals of equal length and values from the table to approximate $\int_1^{1.4} f'(x)\,dx$. Use the approximation for $\int_1^{1.4} f'(x)\,dx$ to estimate the value of $f(1.4)$. Show the computations that lead to your answer.

   (c) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(1.4)$. Show the computations that lead to your answer.

   (d) Write the second-degree Taylor polynomial for f about $x = 1$. Use the Taylor polynomial to approximate $f(1.4)$.

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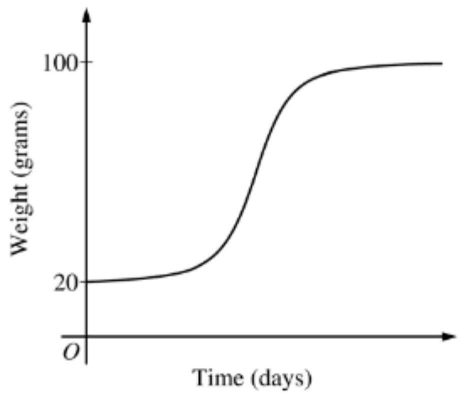
5. The rate at which a baby bird gains weight is proportional to the difference between its adult weight and its current weight. At time $t = 0$, when the bird is first weighed, its weight is 20 grams. If $B(t)$ is the weight of the bird, in grams, at time t days after it is first weighed, then

$$\frac{dB}{dt} = \frac{1}{5}(100 - B).$$

Let $y = B(t)$ be the solution to the differential equation above with initial condition $B(0) = 20$.

(a) Is the bird gaining weight faster when it weighs 40 grams or when it weighs 70 grams? Explain your reasoning.

(b) Find $\frac{d^2B}{dt^2}$ in terms of B. Use $\frac{d^2B}{dt^2}$ to explain why the graph of B cannot resemble the following graph.



(c) Use separation of variables to find $y = B(t)$, the particular solution to the differential equation with initial condition $B(0) = 20$.

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6. The function g has derivatives of all orders, and the Maclaurin series for g is

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+3} = \frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7} - \cdots.$$

(a) Using the ratio test, determine the interval of convergence of the Maclaurin series for g.

(b) The Maclaurin series for g evaluated at $x = \frac{1}{2}$ is an alternating series whose terms decrease in absolute value to 0. The approximation for $g\left(\frac{1}{2}\right)$ using the first two nonzero terms of this series is $\frac{17}{120}$. Show that this approximation differs from $g\left(\frac{1}{2}\right)$ by less than $\frac{1}{200}$.

(c) Write the first three nonzero terms and the general term of the Maclaurin series for $g'(x)$.

Question 1

t (minutes)	0	4	9	15	20
$W(t)$ (degrees Fahrenheit)	55.0	57.1	61.8	67.9	71.0

The temperature of water in a tub at time t is modeled by a strictly increasing, twice-differentiable function W, where $W(t)$ is measured in degrees Fahrenheit and t is measured in minutes. At time $t = 0$, the temperature of the water is 55°F. The water is heated for 30 minutes, beginning at time $t = 0$. Values of $W(t)$ at selected times t for the first 20 minutes are given in the table above.

(a) Use the data in the table to estimate $W'(12)$. Show the computations that lead to your answer. Using correct units, interpret the meaning of your answer in the context of this problem.

(b) Use the data in the table to evaluate $\int_0^{20} W'(t)\,dt$. Using correct units, interpret the meaning of $\int_0^{20} W'(t)\,dt$ in the context of this problem.

(c) For $0 \le t \le 20$, the average temperature of the water in the tub is $\frac{1}{20}\int_0^{20} W(t)\,dt$. Use a left Riemann sum with the four subintervals indicated by the data in the table to approximate $\frac{1}{20}\int_0^{20} W(t)\,dt$. Does this approximation overestimate or underestimate the average temperature of the water over these 20 minutes? Explain your reasoning.

(d) For $20 \le t \le 25$, the function W that models the water temperature has first derivative given by $W'(t) = 0.4\sqrt{t}\cos(0.06t)$. Based on the model, what is the temperature of the water at time $t = 25$?

(a) $W'(12) \approx \dfrac{W(15) - W(9)}{15 - 9} = \dfrac{67.9 - 61.8}{6}$
$= 1.017$ (or 1.016)

The water temperature is increasing at a rate of approximately $1.017\ ^{\circ}$F per minute at time $t = 12$ minutes.

$2: \begin{cases} 1: \text{estimate} \\ 1: \text{interpretation with units} \end{cases}$

(b) $\int_0^{20} W'(t)\,dt = W(20) - W(0) = 71.0 - 55.0 = 16$

The water has warmed by $16\ ^{\circ}$F over the interval from $t = 0$ to $t = 20$ minutes.

$2: \begin{cases} 1: \text{value} \\ 1: \text{interpretation with units} \end{cases}$

(c) $\frac{1}{20}\int_0^{20} W(t)\,dt \approx \frac{1}{20}(4 \cdot W(0) + 5 \cdot W(4) + 6 \cdot W(9) + 5 \cdot W(15))$
$= \frac{1}{20}(4 \cdot 55.0 + 5 \cdot 57.1 + 6 \cdot 61.8 + 5 \cdot 67.9)$
$= \frac{1}{20} \cdot 1215.8 = 60.79$

This approximation is an underestimate, because a left Riemann sum is used and the function W is strictly increasing.

$3: \begin{cases} 1: \text{left Riemann sum} \\ 1: \text{approximation} \\ 1: \text{underestimate with reason} \end{cases}$

(d) $W(25) = 71.0 + \int_{20}^{25} W'(t)\,dt$
$= 71.0 + 2.043155 = 73.043$

$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$

Question 2

For $t \ge 0$, a particle is moving along a curve so that its position at time t is $(x(t), y(t))$. At time $t = 2$, the particle is at position $(1, 5)$. It is known that $\frac{dx}{dt} = \frac{\sqrt{t+2}}{e^t}$ and $\frac{dy}{dt} = \sin^2 t$.

(a) Is the horizontal movement of the particle to the left or to the right at time $t = 2$? Explain your answer. Find the slope of the path of the particle at time $t = 2$.

(b) Find the x-coordinate of the particle's position at time $t = 4$.

(c) Find the speed of the particle at time $t = 4$. Find the acceleration vector of the particle at time $t = 4$.

(d) Find the distance traveled by the particle from time $t = 2$ to $t = 4$.

(a) $\left.\frac{dx}{dt}\right|_{t=2} = \frac{2}{e^2}$

Because $\left.\frac{dx}{dt}\right|_{t=2} > 0$, the particle is moving to the right at time $t = 2$.

$$\left.\frac{dy}{dx}\right|_{t=2} = \frac{dy/dt|_{t=2}}{dx/dt|_{t=2}} = 3.055 \text{ (or 3.054)}$$

$$3: \begin{cases} 1: \text{moving to the right with reason} \\ 1: \text{considers } \frac{dy/dt}{dx/dt} \\ 1: \text{slope at } t = 2 \end{cases}$$

(b) $x(4) = 1 + \int_2^4 \frac{\sqrt{t+2}}{e^t}\,dt = 1.253$ (or 1.252)

$$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$$

(c) Speed $= \sqrt{(x'(4))^2 + (y'(4))^2} = 0.575$ (or 0.574)

Acceleration $= \langle x''(4), y''(4) \rangle$
$= \langle -0.041, 0.989 \rangle$

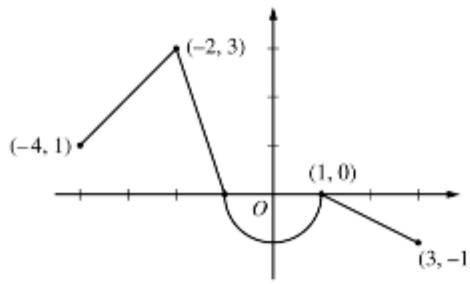
$$2: \begin{cases} 1: \text{speed} \\ 1: \text{acceleration} \end{cases}$$

(d) Distance $= \int_2^4 \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$
$= 0.651$ (or 0.650)

$$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$$

Question 3

Let f be the continuous function defined on $[-4, 3]$ whose graph, consisting of three line segments and a semicircle centered at the origin, is given above. Let g be the function given by $g(x) = \int_1^x f(t)\,dt$.



Graph of f

(a) Find the values of $g(2)$ and $g(-2)$.

(b) For each of $g'(-3)$ and $g''(-3)$, find the value or state that it does not exist.

(c) Find the x-coordinate of each point at which the graph of g has a horizontal tangent line. For each of these points, determine whether g has a relative minimum, relative maximum, or neither a minimum nor a maximum at the point. Justify your answers.

(d) For $-4 < x < 3$, find all values of x for which the graph of g has a point of inflection. Explain your reasoning.

(a) $g(2) = \int_1^2 f(t)\,dt = -\frac{1}{2}(1)\left(\frac{1}{2}\right) = -\frac{1}{4}$

$g(-2) = \int_1^{-2} f(t)\,dt = -\int_{-2}^{1} f(t)\,dt$

$= -\left(\frac{3}{2} - \frac{\pi}{2}\right) = \frac{\pi}{2} - \frac{3}{2}$

$2: \begin{cases} 1: g(2) \\ 1: g(-2) \end{cases}$

(b) $g'(x) = f(x) \Rightarrow g'(-3) = f(-3) = 2$

$g''(x) = f'(x) \Rightarrow g''(-3) = f'(-3) = 1$

$2: \begin{cases} 1: g'(-3) \\ 1: g''(-3) \end{cases}$

(c) The graph of g has a horizontal tangent line where $g'(x) = f(x) = 0$. This occurs at $x = -1$ and $x = 1$.

$g'(x)$ changes sign from positive to negative at $x = -1$. Therefore, g has a relative maximum at $x = -1$.

$g'(x)$ does not change sign at $x = 1$. Therefore, g has neither a relative maximum nor a relative minimum at $x = 1$.

$3: \begin{cases} 1: \text{considers } g'(x) = 0 \\ 1: x = -1 \text{ and } x = 1 \\ 1: \text{answers with justifications} \end{cases}$

(d) The graph of g has a point of inflection at each of $x = -2$, $x = 0$, and $x = 1$ because $g''(x) = f'(x)$ changes sign at each of these values.

$2: \begin{cases} 1: \text{answer} \\ 1: \text{explanation} \end{cases}$

Question 4

x	1	1.1	1.2	1.3	1.4
$f'(x)$	8	10	12	13	14.5

The function f is twice differentiable for $x > 0$ with $f(1) = 15$ and $f''(1) = 20$. Values of f', the derivative of f, are given for selected values of x in the table above.

(a) Write an equation for the line tangent to the graph of f at $x = 1$. Use this line to approximate $f(1.4)$.

(b) Use a midpoint Riemann sum with two subintervals of equal length and values from the table to approximate $\int_1^{1.4} f'(x)\,dx$. Use the approximation for $\int_1^{1.4} f'(x)\,dx$ to estimate the value of $f(1.4)$. Show the computations that lead to your answer.

(c) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(1.4)$. Show the computations that lead to your answer.

(d) Write the second-degree Taylor polynomial for f about $x = 1$. Use the Taylor polynomial to approximate $f(1.4)$.

(a) $f(1) = 15,\ f'(1) = 8$

An equation for the tangent line is
$y = 15 + 8(x - 1)$.

$f(1.4) \approx 15 + 8(1.4 - 1) = 18.2$

$2: \begin{cases} 1: \text{tangent line} \\ 1: \text{approximation} \end{cases}$

(b) $\int_1^{1.4} f'(x)\,dx \approx (0.2)(10) + (0.2)(13) = 4.6$

$f(1.4) = f(1) + \int_1^{1.4} f'(x)\,dx$

$f(1.4) \approx 15 + 4.6 = 19.6$

$3: \begin{cases} 1: \text{midpoint Riemann sum} \\ 1: \text{Fundamental Theorem of Calculus} \\ 1: \text{answer} \end{cases}$

(c) $f(1.2) \approx f(1) + (0.2)(8) = 16.6$

$f(1.4) \approx 16.6 + (0.2)(12) = 19.0$

$2: \begin{cases} 1: \text{Euler's method with two steps} \\ 1: \text{answer} \end{cases}$

(d) $T_2(x) = 15 + 8(x - 1) + \frac{20}{2!}(x - 1)^2$
$\quad = 15 + 8(x - 1) + 10(x - 1)^2$

$f(1.4) \approx 15 + 8(1.4 - 1) + 10(1.4 - 1)^2 = 19.8$

$2: \begin{cases} 1: \text{Taylor polynomial} \\ 1: \text{approximation} \end{cases}$

Question 5

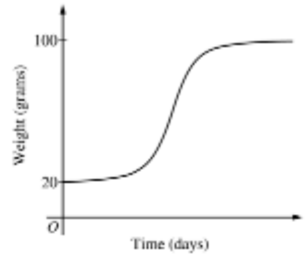
The rate at which a baby bird gains weight is proportional to the difference between its adult weight and its current weight. At time $t = 0$, when the bird is first weighed, its weight is 20 grams. If $B(t)$ is the weight of the bird, in grams, at time t days after it is first weighed, then

$$\frac{dB}{dt} = \frac{1}{5}(100 - B).$$

Let $y = B(t)$ be the solution to the differential equation above with initial condition $B(0) = 20$.

(a) Is the bird gaining weight faster when it weighs 40 grams or when it weighs 70 grams? Explain your reasoning.

(b) Find $\frac{d^2B}{dt^2}$ in terms of B. Use $\frac{d^2B}{dt^2}$ to explain why the graph of B cannot resemble the following graph.



(c) Use separation of variables to find $y = B(t)$, the particular solution to the differential equation with initial condition $B(0) = 20$.

(a) $\left.\frac{dB}{dt}\right|_{B=40} = \frac{1}{5}(60) = 12$

$\left.\frac{dB}{dt}\right|_{B=70} = \frac{1}{5}(30) = 6$

Because $\left.\frac{dB}{dt}\right|_{B=40} > \left.\frac{dB}{dt}\right|_{B=70}$, the bird is gaining weight faster when it weighs 40 grams.

$2: \begin{cases} 1: \text{uses } \frac{dB}{dt} \\ 1: \text{answer with reason} \end{cases}$

(b) $\frac{d^2B}{dt^2} = -\frac{1}{5}\frac{dB}{dt} = -\frac{1}{5}\cdot\frac{1}{5}(100 - B) = -\frac{1}{25}(100 - B)$

Therefore, the graph of B is concave down for $20 \le B < 100$. A portion of the given graph is concave up.

$2: \begin{cases} 1: \frac{d^2B}{dt^2} \text{ in terms of } B \\ 1: \text{explanation} \end{cases}$

(c) $\frac{dB}{dt} = \frac{1}{5}(100 - B)$

$\int \frac{1}{100 - B}\,dB = \int \frac{1}{5}\,dt$

$-\ln|100 - B| = \frac{1}{5}t + C$

Because $20 \le B < 100$, $|100 - B| = 100 - B$.

$-\ln(100 - 20) = \frac{1}{5}(0) + C \Rightarrow -\ln(80) = C$

$100 - B = 80e^{-t/5}$

$B(t) = 100 - 80e^{-t/5}, \quad t \ge 0$

$5: \begin{cases} 1: \text{separation of variables} \\ 1: \text{antiderivatives} \\ 1: \text{constant of integration} \\ 1: \text{uses initial condition} \\ 1: \text{solves for } B \end{cases}$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

Question 6

The function g has derivatives of all orders, and the Maclaurin series for g is
$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+3} = \frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7} - \cdots$.

(a) Using the ratio test, determine the interval of convergence of the Maclaurin series for g.

(b) The Maclaurin series for g evaluated at $x = \frac{1}{2}$ is an alternating series whose terms decrease in absolute value to 0. The approximation for $g\left(\frac{1}{2}\right)$ using the first two nonzero terms of this series is $\frac{17}{120}$. Show that this approximation differs from $g\left(\frac{1}{2}\right)$ by less than $\frac{1}{200}$.

(c) Write the first three nonzero terms and the general term of the Maclaurin series for $g'(x)$.

(a) $\left|\frac{x^{2n+3}}{2n+5} \cdot \frac{2n+3}{x^{2n+1}}\right| = \left(\frac{2n+3}{2n+5}\right) \cdot x^2$

$\lim_{n\to\infty} \left(\frac{2n+3}{2n+5}\right) \cdot x^2 = x^2$

$x^2 < 1 \Rightarrow -1 < x < 1$
The series converges when $-1 < x < 1$.

When $x = -1$, the series is $-\frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$
This series converges by the Alternating Series Test.

When $x = 1$, the series is $\frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \cdots$
This series converges by the Alternating Series Test.

Therefore, the interval of convergence is $-1 \le x \le 1$.

5 :
- 1 : sets up ratio
- 1 : computes limit of ratio
- 1 : identifies interior of interval of convergence
- 1 : considers both endpoints
- 1 : analysis and interval of convergence

(b) $\left|g\left(\frac{1}{2}\right) - \frac{17}{120}\right| < \frac{\left(\frac{1}{2}\right)^5}{7} = \frac{1}{224} < \frac{1}{200}$

2 :
- 1 : uses the third term as an error bound
- 1 : error bound

(c) $g'(x) = \frac{1}{3} - \frac{3}{5}x^2 + \frac{5}{7}x^4 + \cdots + (-1)^n\left(\frac{2n+1}{2n+3}\right)x^{2n} + \cdots$

2 :
- 1 : first three terms
- 1 : general term